ESTIMATION OF A \( k \)-MONOTONE DENSITY, PART 4: LIMIT DISTRIBUTION THEORY AND THE SPLINE CONNECTION

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We study the asymptotic behavior of the Maximum Likelihood and Least Squares estimators of a \( k \)-monotone density \( g_0 \) at a fixed point \( x_0 \) when \( k > 2 \). In Balabdaoui and Wellner (2004a), it was proved that both estimators exist and are splines of degree \( k - 1 \) with simple knots. These knots, which are also the jump points of the \((k - 1)\)st derivative of the estimators, cluster around a point \( x_0 > 0 \) under the assumption that \( g_0 \) has a continuous \( k \)th derivative in a neighborhood of \( x_0 \) and \((-1)^k g_0^{(k)}(x_0) > 0 \). If \( \tau^-_n \) and \( \tau^+_n \) are two successive knots, we prove that the random “gap” \( \tau^+_n - \tau^-_n \) is \( O_p \left( n^{-1/(2k+1)} \right) \) for any \( k > 2 \) if a conjecture about the upper bound on the error in a particular Hermite interpolation via odd-degree splines holds. Based on the order of the gap, the asymptotic distribution of the Maximum Likelihood and Least Squares estimators can be established. We find that the \( j \)th derivative of the estimators at \( x_0 \), converges at the rate \( n^{-(k-j)/(2k+1)} \) for \( j = 0, \ldots, k - 1 \). The limiting distribution depends on an almost surely uniquely defined stochastic process \( H_k \) that stays above (below) the \( k \)-fold integral of Brownian motion plus a deterministic drift, when \( k \) is even (odd). The family of processes \( H_k \) is studied separately in the companion manuscript Balabdaoui and Wellner (2004c).

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1. Introduction. Our interest in the nonparametric estimation of a 
k-monotone density was first motivated by Jewell (1982) on the nonpara-
metric Maximum Likelihood estimator of a scale mixture of exponentials 
g_0,

\begin{equation}
  g_0(x) = \int_0^\infty y^{-1} \exp(-x/y) dF_0(y), \quad x > 0
\end{equation}

where \(F_0\) is any distribution function on \((0, \infty)\). If \(X_1, \ldots, X_n\) are i.i.d. 
with density \(g_0\), Jewell (1982) established that \(\hat{F}_n\), the Maximum Likeli-
hood Estimator (MLE) of the mixing distribution \(F_0\), exists, has at most 
\(n\) support points, and converges weakly to \(F_0\) with probability 1. To our 
knowledge the rates of convergence of either the MLE of the mixing dis-
tribution \(F_0\) or the true mixed density \(g_0\) remain unknown. Jewell (1982) 
noted that the complement of the true cumulative distribution \(1 - G_0\) is the 
Laplace transform of the mixing distribution function \(F_0\), and also the fact 
the class of scale of mixtures of Exponentials given in (1.1) can be identi-
fied as the class of \textit{completely monotone} densities (Bernstein’s theorem). By 
definition, a function \(g\) on \((0, \infty)\) is completely monotone if and only if \(g\) 
is infinitely differentiable on \((0, \infty)\) and \((-1)^kg^{(k)} \geq 0\), for \(k \in \mathbb{R}\) (see e.g. 
Schoenberg (1938), Widder (1941), Williamson (1956), Feller (1971), 
and Gneiting (1998)). If \(g\) is only differentiable up to a finite degree, then it 
is \(k\)-monotone if and only if \((-1)^jg^{(j)}\) is nonnegative, nonincreasing and con-
 convex for \(j = 0, \ldots, k - 2\) if \(k \geq 2\) and simply nonnegative and nonincreasing 
if \(k = 1\) (see e.g. Williamson (1956), Lévy (1962), Gneiting (1999)). The 
class of completely monotone densities is the intersection of all the classes 
of \(k\)-monotone densities, \(k \geq 1\) (Gneiting (1999)) and a completely mono-
tone density can be viewed then as an “\(\infty\)-monotone” density. To prepare 
the ground for establishing the exact rate of convergence of the MLE for 
mixtures of exponentials (or, equivalently, completely monotone densities), 
it therefore seems natural to first establish asymptotic distribution theory 
for the MLE of a \(k\)-monotone density.

When \(k = 1\), the problem specializes to estimating a nonincreasing den-
In this case, the asymptotic distribution theory was established by Prakasa Rao (1969), and revisited by Groeneboom (1985) and Kim and Pollard (1990). They showed that if $x_0$ is a fixed point such that $g'_0(x_0) < 0$ (and assuming that $g'_0$ is continuous in a neighborhood of $x_0$), then the MLE $\hat{g}_n$ (the Grenander estimator), satisfies

$$n^{1/3}(\hat{g}_n(x_0) - g_0(x_0)) \rightarrow_d \left(\frac{1}{2}g_0(x_0)|g'_0(x_0)|\right)^{1/3} 2Z,$$

where $2Z$ is the slope at zero of the greatest convex minorant of two-sided Brownian motion $+t^2$, $t \in \mathbb{R}$. For $k = 2$, Groeneboom, Jongbloed, and Wellner (2001b) considered both the MLE and LSE and established that if the true convex and nonincreasing density $g_0$ satisfies $g''_0(x_0) > 0$ (and assuming that $g''_0$ is continuous in a neighborhood of $x_0$), then

$$\left(\begin{array}{c}
\frac{n^{2/5}}{n^{1/5}} (\bar{g}_n(x_0) - g_0(x_0)) \\
\frac{n^{2/5}}{n^{1/5}} (\bar{g}_n(x_0) - g'(x_0))
\end{array}\right) \rightarrow_d \left(\begin{array}{c}
\left(\frac{1}{24}g_0^2(x_0)g''_0(x_0)\right)^{1/5} H^{(2)}(0) \\
\left(\frac{1}{24}g_0(x_0)g''_0(x_0)^3\right)^{1/5} H^{(3)}(0)
\end{array}\right),$$

where $\bar{g}_n$ is the either the MLE or LSE and $H$ is a random cubic spline function such that $H^{(2)}$ is convex and $H$ stays above the integrated two-sided Brownian motion $+t^4$, $t \in \mathbb{R}$, and touches exactly at those points where $H^{(2)}$ changes its slope (see Groeneboom, Jongbloed, and Wellner (2001a)).

The key result that Groeneboom, Jongbloed, and Wellner (2001b) used to establish (1.3) is that $\tau^{+}_n - \tau^{-}_n = O_p(n^{-1/5})$ as $n \rightarrow \infty$, where $\tau^{-}_n$ and $\tau^{+}_n$ are two successive jump points of the first derivative of $\bar{g}_n$ in the neighborhood of $x_0$. This stochastic order was obtained using the characterizations of the estimators together with the “mid-point property” which we review in Section 2. For $k = 1$, the same property can be used to establish that $n^{-1/3}$ is the order of the gap. As a function of $k$, it is natural to conjecture that $n^{-1/(2k+1)}$ is the general form of the order of the gap. In the problem of nonparametric regression via splines, Mammen and van de Geer (1997) have conjectured that $n^{-1/(2k+1)}$ is the order of the distance between the knot points of their regression spline $\hat{m}$ under the assumption that the true regression curve $m_0$ satisfies our same working assumptions, but the question was left open (see Mammen and van de Geer (1997), page 400). In this
manuscript, we refer to the problem of establishing the order of $\tau_n^+ - \tau_n^-$ as the gap problem.

In Section 2, we show that when $k > 2$, the gap problem is closely related to a “non-classical” Hermite interpolation problem via odd-degree splines. To put the interpolation problem encountered in the next section in context, it is useful to review briefly the related complete Hermite interpolation problem for odd-degree splines which is more “classical” and for which error bounds uniform in the knots are now available. Given a function $f \in C^{(k-1)}[0,1]$ and an increasing sequence $0 = y_0 < y_1 < \cdots < y_m < y_{m+1} = 1$ where $m \geq 1$ is an integer, it is well-known that there exists a unique spline, called the complete spline and denoted here by $Cf$, of degree $2k - 1$ with interior knots $y_1, \ldots, y_m$ that satisfies the $2k + m$ conditions

\[
\begin{align*}
(Cf)(y_i) &= f(y_i), & i &= 1, \ldots, m \\
(Cf)^{(l)}(y_0) &= f^{(l)}(y_0), & (Cf)^{(l)}(y_{m+1}) &= f^{(l)}(y_{m+1}), & l &= 0, \ldots, k - 1;
\end{align*}
\]

see Schoenberg (1963), de Boor (1974), or Nürnberger (1989), page 116, for further discussion. If $j \in \{0, \ldots, k\}$ and $f \in C^{(k+j)}[0,1]$, then there exists $c_{k,j} > 0$ such that

\[
\sup_{0 < y_1 < \cdots < y_m < 1} \|f - Cf\|_{\infty} \leq c_{k,j}\|f^{(k+j)}\|_{\infty}.
\]  

For $j = k$, this “uniform in knots” bound in the complete interpolation problem was first conjectured by de Boor (1973) in (1972) for $k > 4$ as a generalization that goes beyond $k = 2, 3$ and 4 for which the result was already established (see also de Boor (1974)). By a scaling argument, the bound (1.4) implies that, if $f \in C^{(2k)}[a,b], a < b \in \mathbb{R}$, the interpolation error in the complete Hermite interpolation problem is uniformly bounded in the knots, and that the bound is of the order of $(b - a)^{2k}$. One key property of the complete spline interpolant $Cf$ is that $(Cf)^{(k)}$ is the Least Squares approximation of $f^{(k)}$ when $f^{(k)} \in L_2([0,1])$; i.e., if $S_k(y_1, \ldots, y_m)$ denotes the space of splines of order $k$ (degree $k - 1$) and interior knots $y_1, \ldots, y_m$. 
then
\[
\left( \int_0^1 \left( (C^f)^{(k)} - f^{(k)}(x) \right)^2 dx \right) = \min_{S \in S_k(y_1, \ldots, y_m)} \left( \int_0^1 \left( S(x) - f^{(k)}(x) \right)^2 dx \right)
\]
(see e.g. Schoenberg (1963), de Boor (1974), Nürnberger (1989)). Consequently, if \( L_\infty \) denotes the space of bounded functions on \([0, 1]\), then the properly defined map
\[
C^{(k)}[0, 1] \rightarrow S_k(y_1, \ldots, y_m)
\]
\[
f^{(k)} \mapsto (C^f)^{(k)}
\]
is the restriction of the orthoprojector, denoted here by \( P_{S_k(y)} \), where \( y = (y_1, \ldots, y_m) \), from \( L_\infty \) to \( L_\infty \) with respect to the inner product
\[
\langle g, h \rangle = \int_0^1 g(x)h(x)dx.
\]
De Boor (1974) pointed out that, in order to prove the conjecture, it is enough to prove that
\[
\sup_{y} \| P_{S_k(y)} \|_\infty = \sup_{y} \sup_{g} \frac{\| P_{S_k(y)}(g) \|_\infty}{\| g \|}
\]
is bounded, and this was successfully achieved by Shadrin (2001).

The Hermite interpolation problem which arises naturally in Section 2 appears to be another variant of Hermite interpolation problems via odd-degree splines which has not yet been studied in the approximation theory or spline literature. More specifically, if \( f \) is some real-valued function in \( C^{(j)}[0, 1] \) for some \( j \geq 2 \), \( 0 = y_0 < y_1 < \cdots < y_{2k-4} < y_{2k-3} = 1 \) is a given increasing sequence, then there exists a unique spline \( Hf \) of degree \( 2k - 1 \) and interior knots \( y_1, \ldots, y_{2k-4} \) satisfying the \( 4k - 4 \) conditions
\[
(1.6) (Hf)(y_i) = f(y_i), \quad (Hf)'(y_i) = f'(y_i), \quad i = 0, \ldots, 2k - 3.
\]
It turns out that deriving the stochastic order of the distance between two successive knots of the MLE and LSE in the neighborhood of the point of estimation is very closely linked to bounding the error in this new Hermite
interpolation independently of the locations of the knots of the spline interpolant. More precisely, if \( g_t(x) = (x - t)_{k-1}^k/(k - 1)! \) is the power truncated function of degree \( k - 1 \) with unique knot \( t \), then we conjecture that there is a constant \( d_k > 0 \) such that

\[
(1.7)\quad \sup_{t \in (0,1)} \sup_{0 < y_1 < \cdots < y_{2k-4} < 1} \|g_t - Hg_t\|_\infty \leq d_k.
\]

As shown in Balabdaoui and Wellner (2005), the preceding formulation implies that boundedness of the error independently of the knots of the spline interpolant holds true for any \( f \in C^{(k+j)} \). Based on the above conjecture, we can prove that the distance between two consecutive knots in a neighborhood of \( x_0 \) is \( O_p(n^{-1/(2k+1)}) \). In Section 2, we give a statement of our main result: the asymptotic distributions in (1.2) and (1.3) takes the following general form:

\[
\begin{pmatrix}
\frac{n^{\frac{k}{2k+1}}}{k!} (\hat{g}_n(x_0) - g_0(x_0)) \\
\frac{n^{\frac{k-1}{2k+1}}}{k!} (\hat{g}_n^{(1)}(x_0) - g_0^{(1)}(x_0)) \\
\vdots \\
\frac{n^{\frac{1}{2k+1}}}{k!} (\hat{g}_n^{(k-1)}(x_0) - g_0^{(k-1)}(x_0))
\end{pmatrix} \rightarrow d \begin{pmatrix}
c_0(g_0)H_k^{(k)}(0) \\
c_1(g_0)H_k^{(k+1)}(0) \\
\vdots \\
c_{k-1}(g_0)H_k^{(2k-1)}(0)
\end{pmatrix}
\]

where

\[
c_j(g_0) = \left\{ (g_0(x_0))^{k-j} \left( \frac{(-1)^k g_0^{(k)}(x_0)}{k!} \right)^{2j+1} \right\}^{\frac{1}{2k+1}},
\]

for \( j = 0, \ldots, k - 1 \). The rate \( n^{-(k-j)/(2k+1)} \) was found by Balabdaoui and Wellner (2004a) to be the asymptotic minimax lower bound for estimating \( g_0^{(j)}(x_0) \), \( j = 0, \ldots, k - 1 \) under the same working assumptions. The limiting distribution depends on the higher derivatives of \( H_k \), an almost surely uniquely defined process that stays above (below) the \( (k - 1) \)-fold integral of Brownian motion plus the drift \( (k!/(2k)!) \) \( t^{2k} \), when \( k \) is even (odd), and is \( (2k - 2) \)-convex; i.e. the \( 2k - 2 \) derivative of \( H_k \) is convex. The process \( H_k \) is studied separately in Balabdaoui and Wellner (2004c). Proving the existence of \( H_k \) relies also on our conjecture in (1.7) since the key problem,
also referred to as the gap problem, depends on a very similar Hermite interpolation problem, except that the knots of the estimators are replaced by the points of touch between the \((k - 1)\)-fold integral of Brownian motion plus the drift \((k!/(2k)!)) t^{2k} \) and \( H_k \). For more discussion of the background and related problems, see Balabdaoui and Wellner (2004a). For a discussion of algorithms and computational issues, see Balabdaoui and Wellner (2004b).

2. The asymptotic distribution. To prepare for a statement of the main result, we first recall the following theorem from Balabdaoui and Wellner (2004c) giving existence of the processes \( H_k \).

**Theorem 2.1** For all \( k \geq 1 \), let \( Y_k \) denote the stochastic process defined by

\[
Y_k(t) = \begin{cases}
\int_0^t \frac{(t-s)^{k-1}}{(k-1)!}dW(s) + \frac{(-1)^k k! t^{2k}}{(2k)!}, & t \geq 0 \\
\int_0^t \frac{(t-s)^{k-1}}{(k-1)!}dW(s) + \frac{(-1)^k k! t^{2k}}{(2k)!}, & t < 0.
\end{cases}
\]

If Conjecture 3.1 holds (also see the discussion in Balabdaoui and Wellner (2004c)), then there exists an almost surely uniquely defined stochastic process \( H_k \) characterized by the following four conditions:

(i) The process \( H_k \) stays everywhere above the process \( Y_k \):

\[ H_k(t) \geq Y_k(t), \quad t \in \mathbb{R}. \]

(ii) \((-1)^k H_k\) is \(2k\)-convex; i.e. \((-1)^k H_k^{(2k-2)}\) exists and convex.

(iii) The process \( H_k \) satisfies

\[ \int_{-\infty}^\infty (H_k(t) - Y_k(t)) dH_k^{(2k-1)}(t) = 0. \]

(iv) If \( k \) is even, \( \lim_{|t| \to \infty} (H_k^{(2j)}(t) - Y_k^{(2j)}(t)) = 0 \) for \( j = 0, \ldots, (k-2)/2 \);
if \( k \) is odd, \( \lim_{|t| \to \infty} (H_k(t) - Y_k(t)) = 0 \) and \( \lim_{|t| \to \infty} (H_k^{(2j+1)}(t) - Y_k^{(2j+1)}(t)) = 0 \) for \( j = 0, \ldots, (k-3)/2 \).

This is Theorem 2.1 in Balabdaoui and Wellner (2004c).
2.1. The main results and outline of the proofs. Now we are able to state the main result of this paper:

**Theorem 2.2** Let \( x_0 > 0 \) and \( g_0 \) be a \( k \)-monotone density such that \( g_0 \) is \( k \)-times differentiable at \( x_0 \) with \( (-1)^k g_0^{(k)}(x_0) > 0 \) and assume that \( g_0^{(k)} \) is continuous in a neighborhood of \( x_0 \). Let \( \bar{g}_n \) denote either the LSE, \( \tilde{g}_n \) or the MLE \( \hat{g}_n \) and let \( \bar{F}_n \) be the corresponding mixing measure. If Conjecture 3.1 holds, then

\[
\left( \frac{n}{2k+1} (g_0(x_0) - \bar{g}_n(x_0)) \right) \rightarrow_d \left( \begin{array}{c} c_0(g_0)H_k^{(k)}(0) \\ c_1(g_0)H_k^{(k+1)}(0) \\ \vdots \\ c_{k-1}(g_0)H_k^{(2k-1)}(0) \end{array} \right)
\]

and

\[
\frac{n}{2k+1} (\bar{F}_n(x_0) - F(x_0)) \rightarrow_d \frac{(-1)^k x_0^k}{k!} c_{k-1}(g_0)H_k^{(2k-1)}(0)
\]

where

\[
c_j(g_0) = \left\{ \frac{(g_0(x_0))^{k-j} \left( (-1)^k g_0^{(k)}(x_0) \right)^{2j+1}}{k!} \right\}^{1/(2k+1)},
\]

for \( j = 0, \ldots, k - 1 \).

Our proof of Theorem 2.2 proceeds by solving the key gap problem assuming that our Conjecture 3.1 holds. This is carried out in Section 3 in which the main result is:

**Lemma 2.1** Let \( k \geq 2 \) and \( \bar{g}_n \) denote either the LSE \( \tilde{g}_n \) or the MLE \( \hat{g}_n \). If \( g_0 \in \mathcal{D}_k \) satisfies \( g_0^{(k)}(x_0) \neq 0 \) and Conjecture 3.1 holds, then \( \tau_{2k-3} - \tau_0 = O_p(n^{-1/(2k+1)}) \) where \( \tau_0 < \cdots < \tau_{2k-3} \) are \( 2k - 2 \) successive jump points of \( \bar{g}_n^{(k-1)} \) in a neighborhood of \( x_0 \).
Using lemma 2.1 we can establish the rate(s) of convergence of the estimators $\tilde{g}_n$ and $\hat{g}_n$ and their derivatives viewed as local processes in $n^{-1/(2k+1)}$ neighborhoods of the fixed point $x_0$. This is accomplished in Proposition 2.1 (which depends in turn on a preliminary “existence of points” result given in Proposition 4.1). Once the rates have been established, we define localized versions $Y_{n}^{loc}$, $H_{n}^{loc}$, and $\tilde{Y}_{n}^{loc}$, $\tilde{H}_{n}^{loc}$ of certain processes $Y_n$, $H_n$, and $\tilde{Y}_n$, $\tilde{H}_n$ involved in the characterizations of the estimators. The proof then proceeds by showing that:

- The localized processes $Y_{n}^{loc}$ convergence weakly to $Y_{a,\sigma}$ where
  \[
  Y_{a,\sigma}(t) = \begin{cases} 
  \sigma \int_{t}^{0} \int_{s_{k-1}}^{s_{k-1}} \cdots \int_{s_{2}}^{s_{2}} W(s_{1})ds_{1} \cdots ds_{k-1} + a(-1)^{k} \frac{k!}{(2k)!} t^{2k} & t \geq 0 \\
  \sigma \int_{t}^{0} \int_{s_{k-1}}^{s_{k-1}} \cdots \int_{s_{2}}^{s_{2}} W(s_{1})ds_{1} \cdots ds_{k-1} + a(-1)^{k} \frac{k!}{(2k)!} t^{2k} & t \leq 0
  \end{cases}
  \]
  with $\sigma = \sqrt{g(x_0)}$, $a = (-1)^{k} g_{0}^{(k)}(t_0)/k!$ and $W$ a two-sided Brownian motion process starting from 0.

- The localized processes $H_{n}^{loc}$ and $\tilde{H}_{n}^{loc}$ satisfy Fenchel (inequality and equality) relations relative to the localized processes $Y_{n}^{loc}$ and $\tilde{Y}_{n}^{loc}$ respectively.

- We then show via tightness that the localized processes $\tilde{H}_{n}^{loc}$ and $\tilde{H}_{n}^{loc}$ (and all their derivatives up to order $2k-1$) converge to a limit process satisfying the conditions (i) - (iv) of Theorem 2.1, and hence the limit process in both cases is just $H_k$ (up to scaling by constants). When specialized to $t = 0$ this gives the conclusion of Theorem 2.2.

Here is the key rates of convergence proposition.

**Proposition 2.1** Let $x_0 > 0$ and $g_0$ a $k$-monotone density such that $(-1)^{k} g_{0}^{(k)}(x_0) > 0$. Let $\tilde{g}_n$ denote either the MLE $\tilde{g}_n$ or the LSE $\hat{g}_n$. If Conjecture 3.1 holds, then for each $M > 0$ we have,

\[
(2.1) \sup_{|t| \leq M} \left| \tilde{g}_{n}^{(k-1)}(x_0 + n^{-1/(2k+1)} t) - g_{0}^{(k-1)}(x_0) \right| = O_p(n^{-1/(2k+1)})
\]
and

\[
\sup_{|t| \leq M} \left| \hat{g}_n^{(j)}(x_0 + n^{-1/(2k+1)}t) - \sum_{i=j}^{k-1} \frac{n^{-(i-j)/(2k+1)} g_0^{(i)}(x_0) t^{i-j}}{(i-j)!} \right| = O_p(n^{-(k-j)/(2k+1)})
\]

(2.2)

for \( j = 0, \ldots, k-2 \).

Recall that the characterization of the LSE \( \hat{g}_n \) involved the processes \( Y_n \) and \( \tilde{H}_n \) defined by

\[
Y_n(x) = \int_0^x \int_0^{t_k-1} \cdots \int_0^{t_2} \int_0^{t_1} \mathbb{G}_n(t_1) dt_1 dt_2 \cdots dt_k, \quad x \geq 0,
\]

and

\[
\tilde{H}_n(x) = \int_0^x \int_0^{t_k} \cdots \int_0^{t_2} \int_0^{t_1} \tilde{g}_n(t_1) dt_1 dt_2 \cdots dt_k, \quad x \geq 0.
\]

We now define the local \( Y_n \) and \( \tilde{H}_n \)-processes respectively by

\[
Y_{n, loc}(t) = n^{2k+1} \int_{x_0}^{x_0 + t n^{-1/(2k+1)}} \int_{v_k}^{v_{k-1}} \cdots \int_{v_2} \mathbb{G}_n(v_1) - \mathbb{G}_n(x_0) - \int_{x_0}^{v_1} \sum_{j=0}^{k-1} \frac{(u - x_0)^j}{j!} g_0^{(j)}(x_0) du \Pi_{i=1}^{k-1} dv_i,
\]

and

\[
\tilde{H}_{n, loc}(t) = n^{2k+1} \int_{x_0}^{x_0 + t n^{-1/(2k+1)}} \int_{v_k}^{v_{k-1}} \cdots \int_{v_2} \tilde{g}_n(v_1) - \sum_{j=0}^{k-1} \frac{(v_1 - x_0)^j}{j!} g_0^{(j)}(x_0) dv_1 \cdots dv_k
\]

\[ + \tilde{A}_{k-1,n} t^{k-1} + \tilde{A}_{k-2,n} t^{k-2} + \cdots + \tilde{A}_{1,n} t + \tilde{A}_{0,n}, \]

where

\[
\tilde{A}_{j,n} = n^{(2k-j)/(2k+1)} \left( \tilde{H}_n^{(j)}(x_0) - Y_n^{(j)}(x_0) \right), \quad j = 0, \ldots, k-1.
\]
In the following lemma, we will give the asymptotic distribution of the local process \( Y_{n}^{loc} \) in terms of the \((k-1)\)-fold integral of two-sided Brownian motion, \( g_{0}(x_{0}) \), and \( g_{0}^{(k)}(x_{0}) \) assuming that the true density \( g_{0} \) is \( k \)-times continuously differentiable at \( x_{0} \).

**Lemma 2.2** Let \( x_{0} \) be a point where \( g_{0} \) is continuously \( k \)-times differentiable in a neighborhood of \( x_{0} \). Then \( Y_{n}^{loc} \Rightarrow Y_{a,\sigma} \) in \( C[-K,K] \) for each \( K > 0 \) where

\[
Y_{a,\sigma}(t) = \begin{cases} 
\sqrt{g_{0}(x_{0})} \int_{0}^{t} \int_{s_{k-1}}^{s_{k}} \cdots \int_{s_{1}}^{s_{2}} W(s_{1}) ds_{1} \cdots ds_{k-1} + a(-1)^{k} \frac{k!}{2^{k}} t^{2k}, & t \geq 0 \\
\sqrt{g_{0}(x_{0})} \int_{0}^{t} \int_{s_{k-1}}^{s_{k}} \cdots \int_{s_{1}}^{s_{2}} W(s_{1}) ds_{1} \cdots ds_{k-1} + a(-1)^{k} \frac{k!}{2^{k}} t^{2k}, & t < 0
\end{cases}
\]

where \( W \) is standard two-sided Brownian motion starting at \( 0 \), \( \sigma = \sqrt{g_{0}(x_{0})} \), and \( a = (-1)^{k} \frac{g_{0}^{(k)}(x_{0})}{k!} \).

Now consider the MLE \( \hat{g}_{n} \). Recall that the characterization of this estimator involves the process \( \hat{H}_{n} \) given by

\[
\hat{H}_{n}(t) = \int_{0}^{t} \frac{(t-u)^{k-1}}{\hat{g}_{n}(u)} d\bar{G}_{n}(u), \quad \text{for all } t \geq 0
\]

and that

\[
\hat{H}_{n}(t) \begin{cases} 
\leq \frac{t^{k}}{k}, & t \geq 0 \\
= \frac{t^{k}}{k}, & \text{when } t \text{ is a jump point of } \hat{g}_{n}^{(k-1)}
\end{cases}
\]

is a necessary and sufficient condition for \( \hat{g}_{n} \) to be the MLE. Note that \( \hat{H}_{n} \) and \( \hat{H}_{n} \) defined in BALABAOUI AND WELLNER (2004a), Lemma 2.6 are different: \( \hat{H}_{n}(t) = (t^{k}/k)\hat{H}_{n}(t) \) for \( t \geq 0 \). Let \( r_{k} \equiv 1/(2k+1) \). We define the local processes \( \hat{Y}_{n}^{loc} \) and \( \hat{H}_{n}^{loc} \) as

\[
\frac{\hat{Y}_{n}^{loc}(t)}{g_{0}(x_{0})} = n^{2kr_{k}} \int_{x_{0}}^{x_{0}+tn^{-\tau_{k}}} \int_{x_{0}}^{v_{k-1}} \cdots \int_{x_{0}}^{v_{1}} \frac{g_{0}(v) - \sum_{j=0}^{k-1} (v-x_{0})^{j} g_{0}^{(j)}(x_{0})}{\hat{g}_{n}(v)} d\bar{G}_{n}(v) dv_{1} \cdots dv_{k-1}
\]

\[
+ n^{2kr_{k}} \int_{x_{0}}^{x_{0}+tn^{-\tau_{k}}} \int_{x_{0}}^{v_{k-1}} \cdots \int_{x_{0}}^{v_{1}} \frac{1}{\hat{g}_{n}(v)} d(\bar{G}_{n} - G_{0})(v) dv_{1} \cdots dv_{k-1}
\]
and

\[
\frac{\hat{\mathcal{H}}_{\text{loc}}(t)}{g_0(x_0)} = n^{2kr} \int_{x_0}^{x_0+tn^{-r}} \cdots \int_{x_0}^{v_k-1} \int_{x_0}^{v_1} \frac{\hat{g}_n(v) - \sum_{j=0}^{k-1} \frac{(v-x_0)^j}{j!} g_0^{(j)}(x_0)}{\hat{g}_n(v)}
\]

\[dv_1 \cdots dv_{k-1} + A_{(k-1)n}t^{k-1} + \cdots + A_{0n}\]

where

\[
\hat{A}_{jn} = -\frac{n(2k-j)r_k}{(k-1)!j!}g_0(x_0) \left( \hat{H}^{(j)}(x_0) - \frac{(k-1)!}{(k-j)!}x_0^{k-j} \right), \quad j = 0, \ldots, k-1.
\]

**Lemma 2.3** Let \( K > 0 \). Then \( \hat{\mathcal{Y}}_{\text{loc}} \Rightarrow Y_{a,\sigma} \) in \( C[-K,K] \).

Now let \( \hat{\mathcal{H}}_{\text{loc}}^{(j)} \) and \( \hat{\mathcal{Y}}_{\text{loc}}^{(j)} \) denote either \( \hat{H}_{\text{loc}}^{(j)} \) or \( \hat{H}_{\text{loc}}^{(j)} \) and either \( \mathcal{Y}_{\text{loc}}^{(j)} \) or \( \hat{\mathcal{Y}}_{\text{loc}}^{(j)} \) respectively.

**Lemma 2.4** The localized processes \( \hat{\mathcal{Y}}_{\text{loc}}^{(j)} \) and \( \hat{\mathcal{H}}_{\text{loc}}^{(j)} \), satisfy

\[
\hat{H}_{\text{loc}}^{(j)}(t) - \hat{\mathcal{Y}}_{\text{loc}}^{(j)}(t) \geq 0 \quad \text{for all} \quad t \geq 0,
\]

with equality if \( x_0 + tn^{-1/(2k+1)} \) is a jump point of \( \hat{g}_n^{(k-1)} \).

**Lemma 2.5** The limit process \( Y_{a,\sigma} \) in Lemma 2.2 and Lemma 2.3 satisfies

\[
Y_{a,\sigma}(t) \overset{d}{=} \frac{1}{s_1} Y_k \left( \frac{t}{s_2} \right)
\]

where \( Y_k \equiv Y_{1,1} \) and

\[
s_1 = \frac{1}{\sqrt{g_0(x_0)}} \left( \frac{(-1)^k g_0^{(k)}(x_0)}{k! \sqrt{g_0(x_0)}} \right)^{(2k-1)/(2k+1)}
\]

\[
s_2 = \left( \frac{\sqrt{g_0(x_0)}}{(-1)^k g_0^{(k)}(x_0)} \right)^{2/(2k+1)}.
\]

To show that the derivatives of \( \hat{H}_{\text{loc}}^{(j)} \) are tight, we need the following lemma.
Lemma 2.6 For all \( j \in \{0, \ldots, k-1\} \), let \( \bar{A}_{jn} \) denote either \( \bar{A}_j \) or \( \tilde{A}_{jn} \).

If Conjecture 3.1 holds, then

(2.5) \[ \bar{A}_{jn} = O_p(1). \]

Now we rescale the processes \( Y_n^{loc}, \bar{H}_n^{loc} \), and \( \tilde{Y}_n^{loc}, \tilde{H}_n^{loc} \) so that the rescaled \( Y_n^{loc} \) converges to the canonical limit process \( Y_k \) defined in Lemma 2.5. Since the scaling of \( Y_n^{loc} \) will be exactly the same as the one we used for \( Y_k \), we define \( \tilde{H}_n^l \) and \( \tilde{Y}_n^l \) by

\[
\tilde{H}_n^l(t) = s_1 \bar{H}_n^{loc}(s_2t), \quad \tilde{Y}_n^l(t) = s_1 \tilde{Y}_n^{loc}(s_2t)
\]

where \( s_1 \) and \( s_2 \) are given by (2.3) and (2.4) respectively. Now, we can write for the least squares case, \( \bar{H}_n^l = \tilde{H}_n^l \),

\[
\begin{align*}
(\bar{H}_n^l)^{(k)}(0) &= s_1 s_2^k (\bar{H}_n^{loc})^{(k)}(0) = n^{k/(2k+1)} c_k(g_0)(\tilde{g}_n(x_0) - g_0(x_0)) \\
(\bar{H}_n^l)^{(k+1)}(0) &= s_1 s_2^{k+1} (\bar{H}_n^{loc})^{(k+1)}(0) = n^{(k-1)/(2k+1)} c_{k-1}(g_0)(\tilde{g}_n(x_0) - g_0(x_0)) \\
(\bar{H}_n^l)^{(k+2)}(0) &= s_1 s_2^{k+2} (\bar{H}_n^{loc})^{(k+2)}(0) = n^{(k-2)/(2k+1)} c_{k-2}(g_0)(\tilde{g}_n''(x_0) - g_0''(x_0)) \\
&\quad \vdots \\
(\bar{H}_n^l)^{(2k-1)}(0) &= s_1 s_2^{2k-1} (\bar{H}_n^{loc})^{(2k-1)}(0) \\
&= n^{1/(2k+1)} c_1(g_0)(\tilde{g}_n^{(k-1)}(x_0) - g_0^{(k-1)}(x_0)).
\end{align*}
\]

In the maximum likelihood case, \( \bar{H}_n^l = \tilde{H}_n^l \), we have

\[
\begin{align*}
(\tilde{H}_n^l)^{(k)}(0) &= n^{k/(2k+1)} c_k(g_0) g_0(x_0) \frac{(\tilde{g}_n(x_0) - g_0(x_0))}{\tilde{g}_n(x_0)} \\
(\tilde{H}_n^l)^{(k+1)}(0) &= n^{(k-1)/(2k+1)} c_{k-1}(g_0) g_0(x_0) \left( \frac{g_n'(x_0) - g_0'(x_0)}{\tilde{g}_n(x_0)} \right) \\
&\quad + (\tilde{g}_n(x_0) - g_0(x_0)) \left( \frac{1}{\tilde{g}_n(x)} \right)'_{x=x_0} \\
&= n^{(k-1)/(2k+1)} c_{k-1}(g_0) g_0(x_0) \left( \frac{g_n'(x_0) - g_0'(x_0)}{\tilde{g}_n(x_0)} \right).
\end{align*}
\]
\[
- \frac{\hat{g}_n(x_0) - g_0(x_0)}{(\hat{g}_n(x_0))^2} \hat{g}_n'(x_0)
\]

\[
\vdots
\]

\[
(\tilde{H}_n^l)^{(2k-1)}(0) = n^{1/(2k+1)} c_1(g_0) g_0(x_0) \left( \frac{(\hat{g}_n^{(k-1)}(x_0) - g_0^{(k-1)}(x_0))}{\hat{g}_n(x_0)} \right) + \sum_{l=0}^{k-2} \binom{k-1}{l} \left( \hat{g}_n^{(l)}(x_0) - g_0^{(l)}(x_0) \right) \left( \frac{1}{g_0(x)} \right)_{x=x_0}^{(k-1-l)}
\]

Note that the factor \(g_0(x_0)/\hat{g}_n(x_0)\) converges in probability to 1. Moreover, for \(j = 1, \ldots, k - 1\) it can be shown, using Proposition 2.1 and uniform consistency of \(\hat{g}_n^{(j)}\) in the neighborhood of \(x_0\), that the second terms in the above expressions converge to 0 in probability, and hence

\[
|\left(\tilde{H}_n^l\right)^{(k+j)}(0) - n^{(k-j)/(2k+1)} c_{k-j}(g_0)(\hat{g}_n^{(j)}(x_0) - g_0^{(j)}(x_0))| \to_p 0
\]

for \(j = 0, \ldots, k - 1\).

**Lemma 2.7** Let \(c > 0\). Then

\[
((\tilde{H}_n^l)^{(0)}, (\tilde{H}_n^l)^{(1)}, \ldots, (\tilde{H}_n^l)^{(2k-1)}) \Rightarrow (H_k^{(0)}, H_k^{(1)}, \ldots, H_k^{(2k-1)})
\]

in \((D[-c, c])^{2k}\) where \(H_k\) is the stochastic process defined in Theorem 2.1.

### 3. The gap problem: the order is \(n^{-1/(2k+1)}\).

Recall that it was assumed that \(g_0\) is \(k\)-times continuously differentiable at \(x_0\) and that \((-1)^k g_0^{(k)}(x_0) > 0\). Under a weaker assumption, BALABAOUİ AND WELLNER (2004A) proved strong consistency of the \((k - 1)\)-st derivative of the MLE and LSE. This implies that the number of jump points of this derivative, in a small neighborhood of \(x_0\), diverges to infinity almost surely as the sample size \(n \to \infty\). This “clustering” phenomenon is one of the most crucial elements in studying the local asymptotics of the estimators. The jump points form then a sequence that converges to \(x_0\) almost surely and therefore the distance between two successive jump points, for example located just before and
after \( x_0 \), converges to 0 as \( n \to \infty \). But it is not enough to know that the “gap” between these points converges to 0: an upper bound for this rate of convergence is needed.

**Lemma 3.1** Let \( k \geq 2 \) and \( \tilde{g}_n \) denote either the LSE \( \tilde{g}_n \) or the MLE \( \hat{g}_n \). If \( g_0 \in D_k \) satisfies \( g_0^{(k)}(x_0) \neq 0 \) and Conjecture 3.1 holds, then \( \tau_{2k-3} - \tau_0 = O_p(n^{-1/(2k+1)}) \) where \( \tau_0 < \cdots < \tau_{2k-3} \) are \( 2k - 2 \) successive jump points of \( \tilde{g}_n^{(k-1)} \) in a neighborhood of \( x_0 \).

In the following subsection, we describe the difficulties of establishing this result for \( k > 2 \). In the general case, the problem becomes more difficult than the problem in the special case \( k = 2 \).

3.1. **Fundamental differences.** Let \( \tau^-_n \) and \( \tau^+_n \) be the last and first jump points of the \((k-1)\)-st derivative of either the MLE or LSE, located before and after \( x_0 \) respectively. To obtain a better understanding of the gap problem, we describe the reasoning used by Groeneboom, Jongbloed, and Wellner (2001b) in order to prove that \( \tau^+_n - \tau^-_n = O_p(n^{-1/5}) \) for the special case \( k = 2 \). Here, we restrict ourselves only to the LSE since it is a simpler case to deal with than the MLE. Recall that for \( k = 2 \) the characterization of the LSE, \( \tilde{g}_n \), is given by

\[
\tilde{H}_n(x) \begin{cases} 
\geq Y_n(x), & x \geq 0 \\
= Y_n(x), & \text{if } x \text{ is a jump point of } \tilde{g}'_n
\end{cases}
\]

where

\[
\tilde{H}_n(x) = \int_0^x (x-t)\tilde{g}_n(t)dt, \quad \text{and} \quad Y_n(x) = \int_0^x (x-t)d\mathbb{G}_n(t),
\]

and \( \mathbb{G}_n \) is the empirical distribution function. For ease of notation, we omit writing the subscript \( n \) on the jump points, but their dependence on \( n \) should be kept in mind. On the interval \( [\tau^-, \tau^+] \), the function \( \tilde{g}'_n \) is constant since they are no more jump points in this interval. This implies that \( \tilde{H}_n \) is
polynomial of degree 3 on \([\tau^-, \tau^+]\). But, from the characterization in (3.1), it follows that

\[
\tilde{H}_n(\tau^\pm) = Y_n(\tau^\pm), \quad \tilde{H}'_n(\tau^\pm) = Y'_n(\tau^\pm).
\]

These four boundary conditions allow us to fully determine the cubic polynomial \(\tilde{H}_n\) on \([\tau^-, \tau^+]\). Using the explicit expression for \(\tilde{H}_n\) and evaluating it at the mid-point \(\bar{\tau} = (\tau^- + \tau^+)/2\), Groeneboom, Jongbloed, and Wellner (2001b) established that

\[
\tilde{H}_n(\bar{\tau}) = \frac{Y_n(\tau^-) + Y_n(\tau^+)}{2} - \frac{(G_n(\tau^+) - G_n(\tau^-))(\tau^+ - \tau^-)}{8}.
\]

Groeneboom, Jongbloed and Wellner refer to this as the “mid-point property”. By applying the first condition (the inequality condition) in (3.1), it follows that

\[
\frac{Y_n(\tau^-) + Y_n(\tau^+)}{2} - \frac{(G_n(\tau^+) - G_n(\tau^-))(\tau^+ - \tau^-)}{8} \geq Y_n(\bar{\tau}).
\]

The inequality in the last display can be rewritten as

\[
\frac{Y_0(\tau^-) + Y_0(\tau^+)}{2} - \frac{(G_0(\tau^+) - G_0(\tau^-))(\tau^+ - \tau^-)}{8} \geq E_n
\]

where \(G_0\) and \(Y_0\) are the true counterparts of \(G_n\) and \(Y_n\) respectively, and \(E_n\) a random error. Using empirical process theory, Groeneboom, Jongbloed, and Wellner (2001b) showed that

\[
(3.2) \quad |E_n| = O_p(n^{-4/5}) + o_p((\tau^+ - \tau^-)^4).
\]

On the other hand, Groeneboom, Jongbloed, and Wellner (2001b) established that there exists a universal constant \(C > 0\) such that

\[
(3.3) \quad \frac{Y_0(\tau^-) + Y_0(\tau^+)}{2} - \frac{(G_0(\tau^+) - G_0(\tau^-))(\tau^+ - \tau^-)}{8} = -Cg''(x_0)(\tau^+ - \tau^-)^4 + o_p((\tau^+ - \tau^-)^4).
\]

Combining the results in (3.2) and (3.3), it follows that

\[
\tau^+ - \tau^- = O_p(n^{-1/5}).
\]
The problem has two main features that make the above arguments work. First of all, the polynomial \( \tilde{H}_n \) can be fully determined on \([\tau^-, \tau^+]\) and therefore it can be evaluated at any point between \( \tau^- \) and \( \tau^+ \). Second of all, it can be expressed via the empirical process \( \mathbb{Y}_n \) and that enables us to “get rid of” terms depending on \( \tilde{g}_n \) whose rate of convergence is still unknown at this stage. We should also add that the problem is symmetric around \( \bar{\tau} \), a property that helps establishing the formula derived in (3.3). When \( k > 2 \), we have established in Balabdaoui and Wellner (2004a), Proposition 2.2, that \( \tilde{g}_n \) is the LSE if and only if

\[
(3.4) \quad \tilde{H}_n(x) \begin{cases} 
\geq \mathbb{Y}_n(x), & x \geq 0 \\
= \mathbb{Y}_n(x), & \text{if } x \text{ is a jump point of } \tilde{g}_n^{(k-1)} 
\end{cases}
\]

where

\[
\tilde{H}_n(x) = \int_0^x \frac{(x-t)^{k-1}}{(k-1)!} \tilde{g}_n(t) dt
\]

and

\[
\mathbb{Y}_n(x) = \int_0^x \frac{(x-t)^{k-1}}{(k-1)!} dG_n(t).
\]

If \( \tau^- \), \( \tau^+ \) are two successive jump points of \( \tilde{g}_n^{(k-1)} \), then the four equalities

\[
\tilde{H}_n(\tau^\pm) = \mathbb{Y}_n(\tau^\pm), \quad \text{and} \quad \tilde{H}'_n(\tau^\pm) = \mathbb{Y}'_n(\tau^\pm)
\]

still hold. However, these equations are not enough to determine the polynomial \( \tilde{H}_n \), now of degree \( 2k - 1 \), on the interval \([\tau^-, \tau^+]\) when \( k > 2 \). One would need \( 2k \) conditions to be able to achieve this. [We would be in this situation if we had equality of the higher derivatives of \( \tilde{H}_n \) and \( \mathbb{Y}_n \) at \( \tau^- \) and \( \tau^+ \), that is

\[
(3.5) \quad \tilde{H}_n^{(j)}(\tau^-) = \mathbb{Y}_n^{(j)}(\tau^-), \quad \tilde{H}_n^{(j)}(\tau^+) = \mathbb{Y}_n^{(j)}(\tau^+)
\]

for \( j = 0, \ldots, k - 1 \), but the characterization (3.4) does not give this much.] Thus it becomes clear that two jump points are not sufficient to determine
the piecewise polynomial $\tilde{H}_n$. However, if we consider $p > 2$ jump points $\tau_0 < \cdots < \tau_{p-1}$ (all located e.g. after $x_0$), $\tilde{H}_n$ is a spline of degree $2k - 1$; that is, $\tilde{H}_n$ is a polynomial of degree $2k - 1$ on $(\tau_j, \tau_{j+1})$ for $j = 0, \ldots, p-2$ and is $(2k - 2)$-times differentiable at its knot points $\tau_0, \ldots, \tau_{p-1}$. In the next subsection, we prove that if $p = 2k - 2$, the spline $\tilde{H}_n$ is completely determined on $[\tau_0, \tau_{2k-3}]$ by the conditions

\begin{equation}
\tilde{H}_n(\tau_i) = \mathbb{Y}(\tau_i), \quad \tilde{H}_n'(\tau_i) = \mathbb{Y}'(\tau_i), \quad i = 0, \ldots, 2k - 3.
\end{equation}

This result proves to be very useful for determining the stochastic order of the distance between two successive jump points in a small neighborhood of $x_0$ if our Conjecture 3.1 on the uniform boundedness of the error in the “non-classical” Hermite interpolation problem via splines of odd-degree defined in (1.6) holds.

### 3.2. A Hermite interpolation problem

In the next lemma, we prove that given $2k - 2$ successive jump points $\tau_0 < \cdots < \tau_{2k-3}$ of $\tilde{g}_n^{(k-1)}$, $\tilde{H}_n$ is the unique solution of the Hermite problem given by (3.6). But before that, we need the following lemma which gives a definition of B-splines.

**Lemma 3.2** Let $m \geq 1$ be an integer and $x_1 < \cdots < x_{m+1}$ be arbitrary $(m + 1)$ points in $\mathbb{R}$. There exists a unique vector $(a_1, \ldots, a_{m+1}) \in \mathbb{R}^{m+1}$ such that the spline

\[
B(t) = \sum_{i=1}^{m+1} a_i (t - x_i)^{m-1}, \quad t \in \mathbb{R}
\]

satisfies

\begin{align}
B(t) &= 0, \quad \text{if } t \leq x_1 \text{ or } t \geq x_{m+1} \\
B_k(t) &= 0, \quad \text{if } t \in (x_1, x_{m+1}) \\
\int_{x_1}^{x_{m+1}} B(t)dt &= 1.
\end{align}

$B$ is called the $B$-spline of degree $m-1$ with support $[x_1, x_{m+1}]$. Furthermore,

\begin{equation}
B(t) = [x_1, \ldots, x_{m+1}](t - \cdot)^{m-1}, \quad t \in \mathbb{R};
\end{equation}
thus $B(t)$ is the divided difference of order $m$ of the function $x \mapsto (-1)^m m(t - x)_+^{m-1}$, $x \in \mathbb{R}$ with respect to the knots $x_1, \ldots, x_{m+1}$.

**Proof.** See e.g. Nürnberg (1989), Theorems 2.2 and 2.9, pages 96 and 99.

**Remark 3.1** Note that for any $a$ and $b$ in $\mathbb{R}$, we have

$$(b - a)^{m-1} = (b - a)_+^{m-1} + (-1)^{m-1} (a - b)_+^{m-1}.$$ 

On the other hand, we can write

$$
\sum_{i=1}^{m+1} a_i (t - x_i)^{m-1} = \sum_{i=1}^{m+1} a_i \sum_{l=0}^{m-1} \binom{m-1}{l} x_i^l t^{m-1-l} \\
= \sum_{l=0}^{m-1} \binom{m-1}{l} \left( \sum_{i=1}^{m+1} a_i x_i^l \right) t^{m-1-l} = 0, \quad \text{for } t \in \mathbb{R},
$$

where the last equality follows from the identities in (2.4) of Theorem 2.2 in Nürnberg (1989). Therefore, $B$ can also be given by

$$B(t) = (-1)^m \sum_{i=1}^{m+1} a_i (x_i - t)_+^{m-1} \quad t \in \mathbb{R},$$

or equivalently

$$(3.11) \quad B(t) = [x_1, \ldots, x_{m+1}] m(- t)_+^{m-1}.$$ 

The latter form will be used in the rest of this section.

**Lemma 3.3** Let $k \geq 2$. The function $\hat{H}_n$ characterized by (3.4) is a spline of degree $2k - 1$. Moreover, given any $2k - 2$ successive jump points of $\hat{H}_n^{(2k-1)}$, $\tau_0 < \ldots < \tau_{2k-3}$, the $(2k - 1)$-th spline $\hat{H}_n$ is uniquely determined on $[\tau_0, \tau_{2k-3}]$ by the values of the process $\mathcal{Y}_n$ and of its derivative $\mathcal{Y}_n'$ at $\tau_0, \ldots, \tau_{2k-3}$. Furthermore, for any arbitrary points $\tau_{-(2k-1)} < \cdots < \tau_{-1}$ to
the left of $\tau_0$ and $\tau_{2k-2} < \cdots < \tau_{4k-4}$ to the right of $\tau_{2k-3}$, there exist coefficients $\alpha_{-(2k-1)}, \ldots, \alpha_{2k-4}$ depending on $Y_n(\tau_i)$ and $Y'_n(\tau_i)$, $i = 0, \ldots, 2k-3$, such that the spline $\tilde{H}_n$ can be written as

$$
(3.12) \quad \tilde{H}_n(t) = \sum_{i=-(2k-1)}^{2k-4} \alpha_i B_i(t),
$$

for all $t \in [\tau_0, \tau_{2k-3}]$ where, for $i = -(2k-1), \ldots, 2k-4$, $B_i$ is the B-spline of degree $2k-1$ corresponding to the set of knots $\{\tau_i, \ldots, \tau_{i+2k}\}$.

**Proof.** We know that for any jump point $\tau$ of $\tilde{H}_n^{(2k-1)}$, we have

$$
\tilde{H}_n(\tau) = Y_n(\tau) \quad \text{and} \quad \tilde{H}'_n(\tau) = Y'_n(\tau).
$$

This can viewed as a Hermite interpolation problem if we consider that the interpolated function is the process $Y_n$ and that the interpolating spline is $\tilde{H}_n$ (see e.g. Nürenberger (1989), Definition 3.6, pages 108 and 109). Existence and uniqueness of the spline interpolant follows easily from the Schoenberg-Whitney-Karlin-Ziegler Theorem (Schoenberg and Whitney (1953); Theorem 3, page 529, Karlin and Ziegler (1966); or see Theorem 3.7, page 109, Nürenberger (1989); or Theorem 9.2, page 162, DeVore and Lorentz (1993)). If we choose to write $\tilde{H}_n$ in the B-spline basis, then we can find $\alpha_{-(2k-1)}, \ldots, \alpha_{2k-4}$ such that

$$
\tilde{H}_n(t) = \sum_{i=-(2k-1)}^{2k-4} \alpha_i B_i(t),
$$

for all $t \in [a, b] \equiv [\tau_0, \tau_{2k-3}]$, where $\alpha^T = (\alpha_{-(2k-1)}, \ldots, \alpha_{2k-4})$ is the unique solution of the linear system

$$
(3.13) \quad M \alpha = \begin{pmatrix} Y_n(\tau_0) \\ Y'_n(\tau_0) \\ \vdots \\ Y_n(\tau_{2k-3}) \\ Y'_n(\tau_{2k-3}) \end{pmatrix}
$$
where
\[
M \equiv \begin{pmatrix}
B_{-(2k-1)}(\tau_0) & \cdots & B_{2k-4}(\tau_0) \\
(B_{-(2k-1)})'(\tau_0) & \cdots & (B_{2k-4})'(\tau_0) \\
\vdots & \ddots & \vdots \\
B_{-(2k-1)}(\tau_{2k-3}) & \cdots & B_{2k-4}(\tau_{2k-3}) \\
(B_{-(2k-1)})'(\tau_{2k-3}) & \cdots & (B_{2k-4})'(\tau_{2k-3})
\end{pmatrix}
\]
and \(B_i, i = -(2k-1), \ldots, 2k-4\), are \((4k-4)\) linearly independent B-splines of degree \(2k-1\) and knots \(\tau_i < \cdots < \tau_{i+2k}\).

In the following lemma, we prove a preparatory result that will be used later for deriving the stochastic order of the distance between successive knots of the estimators in a neighborhood of \(x_0\).

**Lemma 3.4** Let \(\bar{\tau} \in \bigcup_{i=0}^{2k-4} (\tau_i, \tau_{i+1})\). If \(e_k(x)\) denotes the error at \(x\) of the Hermite interpolation of the function \(x^{2k}/(2k)!\) at the points \(\tau_0, \ldots, \tau_{2k-3}\), then
\[
(3.14) \quad g_0^{(k)}(\bar{\tau})e_k(\bar{\tau}) \leq E_n + R_n
\]
where \(E_n\) defined in (3.16) is a random error and \(R_n\) defined in (3.18) is a remainder that both depend on the knots \(\tau_0, \ldots, \tau_{2k-3}\) and the point \(\bar{\tau}\).

**Proof.** In this proof, we use the explicit B-spline representation of \(\tilde{H}_n\) that was introduced in the previous lemma. Let \(A = (a_{ij})_{ij}\) and \(B = (b_{ij})_{ij}\) be the \((4k-4) \times (k-1)\) sub-matrices obtained by extracting the odd and even columns of the inverse of the matrix \(M\) given in (3.13). We can write,
\[
\tilde{H}_n(x) = \sum_{i = -(2k-1)}^{2k-4} \left( \sum_{j=0}^{2k-3} (a_{ij}Y_n(\tau_j) + b_{ij}Y'_n(\tau_j)) \right) B_i(x)
\]
for all \(x \in [\tau_0, \tau_{2k-3}]\). Fix \(x = \bar{\tau} \in \bigcup_{i=0}^{2k-4} (\tau_i, \tau_{i+1})\). From the inequality condition in the characterization of the LSE, it follows that
\[
\sum_{i = -(2k-1)}^{2k-4} \left( \sum_{j=0}^{2k-3} (a_{ij}Y_n(\tau_j) + b_{ij}Y'_n(\tau_j)) \right) B_i(\bar{\tau}) \geq Y_n(\bar{\tau})
\]
or equivalently

\begin{equation}
\sum_{i=-(2k-1)}^{2k-4} \left( \sum_{j=0}^{2k-3} (a_{ij} Y_0(\tau_j) + b_{ij} Y_0'(\tau_j)) \right) B_i(\bar{\tau}) - Y_0(\bar{\tau}) \geq -\mathbb{E}_n
\end{equation}

where \( Y_0 \) is the \( k \)-fold integral of the true density \( g_0 \) and \( \mathbb{E}_n \) is given by

\begin{equation}
\mathbb{E}_n = \sum_{i=-(2k-1)}^{2k-4} \left( \sum_{j=0}^{2k-3} (a_{ij}(\mathbb{Y}_n - Y_0) - b_{ij}(\mathbb{Y}_n' - Y_0') - (\tau_j)) \right) B_i(\bar{\tau}) + \mathbb{Y}_0(\bar{\tau}) - \mathbb{Y}_n(\bar{\tau}).
\end{equation}

Based on the working assumptions, the function \( Y_0 \) is \( (2k) \)-times continuously differentiable in a small neighborhood of \( x_0 \). Using Taylor expansion of \( Y_0(\tau_j) \) and \( Y_0'(\tau_j) \) around \( \bar{\tau} \) up to the orders \( 2k \) and \( 2k - 1 \) respectively, the inequality in (3.15) can be rewritten as

\begin{equation}
\sum_{i=-(2k-1)}^{2k-4} \left( \sum_{j=0}^{2k-3} a_{ij} B_i(\bar{\tau}) \right) Y_0(\bar{\tau}) + \sum_{i=-(2k-1)}^{2k-4} \left( \sum_{j=0}^{2k-3} a_{ij}(\tau_j - \bar{\tau}) B_i(\bar{\tau}) \right) Y_0'(\bar{\tau}) + \cdots + \sum_{i=-(2k-1)}^{2k-4} \left( \sum_{j=0}^{2k-3} a_{ij}(\tau_j - \bar{\tau})^{2k-1} B_i(\bar{\tau}) \right) Y_0''(\bar{\tau}) + \mathbb{R}_n
\end{equation}

where \( \mathbb{R}_n \) is a remainder that can be given in the integral form

\begin{equation}
\sum_{i=-(2k-1)}^{2k-4} \left( \sum_{j=0}^{2k-3} a_{ij} \int_{\bar{\tau}}^{\tau_j} \frac{(\tau_j - t)^{2k-1}}{(2k-1)!} (g_0^{(k)}(t) - g_0^{(k)}(\bar{\tau})) dt + b_{ij} \int_{\bar{\tau}}^{\tau_j} \frac{(\tau_j - t)^{2k-2}}{(2k-2)!} (g_0^{(k)}(t) - g_0^{(k)}(\bar{\tau})) dt \right) B_i(\bar{\tau}).
\end{equation}
The remainder $R_n$ can be viewed as the Hermite interpolant of the function
\[ x \mapsto \int_{\bar{\tau}}^{x} \frac{(x - t)^{2k-1}}{(2k - 1)!} (g^{(k)}_0(t) - g^{(k)}_0(\bar{\tau}))dt \]
at the point $\bar{\tau}$. The order of $R_n$ will be determined in the next subsection. Note that
\begin{equation}
\sum_{i = -(2k-1)}^{2k-4} \sum_{j=0}^{2k-3} a_{ij} \left( \sum_{i = -(2k-1)}^{2k-4} \sum_{j=0}^{2k-3} a_{ij} (\tau_j - \bar{\tau}) + b_{ij} \right) B_i(\bar{\tau}) = 0
\end{equation}

Indeed, since the space of splines of degree $2k - 1$ and with simple knots $\tau_0, \ldots, \tau_{2k-3}$ includes all the polynomials of degree $\leq 2k - 1$, the solution of the Hermite problem when the interpolated function is a polynomial of degree $\leq 2k - 1$ is the polynomial itself. Therefore, if we consider $P_0(t) = 1, P_1(t) = t - \bar{\tau}, \ldots, P_{2k-1}(t) = (t - \bar{\tau})^{2k-1}/(2k - 1)!$, the previous terms are identically zero since they are exactly equal to $P_j(\bar{\tau}) = 0, j = 0, \ldots, 2k - 1$. Now
\begin{equation}
\sum_{i = -(2k-1)}^{2k-4} \sum_{j=0}^{2k-3} \left( a_{ij} \frac{(\tau_j - \bar{\tau})^{2k}}{(2k)!} + b_{ij} \frac{(\tau_j - \bar{\tau})^{2k-1}}{(2k - 1)!} \right) B_i(\bar{\tau})
\end{equation}
can be recognized as the Hermite interpolant of $x \mapsto (x - \bar{\tau})^{2k}/(2k)!$ at the point $\bar{\tau}$ with interpolation at the knots $\tau_0, \ldots, \tau_{2k-3}$. But it turns out that this interpolating function is equal to $e_k(\bar{\tau})$, the error of Hermite interpolation of the function $x^{2k}/(2k)!$ evaluated at $\bar{\tau}$. To see this, we write, using the binomial identity,
\[
\begin{align*}
&= \sum_{i=-(2k-1)}^{2k-4} \left( \sum_{j=0}^{2k-3} a_{ij} \frac{(\tau_j)^{2k}}{(2k)!} + b_{ij} \frac{(\tau_j)^{2k-1}}{(2k-1)!} \right) B_i(\bar{\tau}) \\
&\quad + \sum_{r=1}^{2k-1} \left( \sum_{i=-(2k-1)}^{2k-4} \left( \sum_{j=0}^{2k-3} a_{ij} \frac{(\tau_j)^{2k-r}}{(2k)!} \right) \right) \sum_{j=0}^{2k-3} \frac{(\tau_j)^{2k-r}}{(2k)!} B_i(\bar{\tau}) (-1)^r \bar{\tau}^r \\
&\quad + \left( \sum_{i=-(2k-1)}^{2k-4} \left( \sum_{j=0}^{2k-3} a_{ij} \right) B_i(\bar{\tau}) \right) \bar{\tau}^{2k-2k}.
\end{align*}
\]

Using the identity
\[
\binom{2k-1}{r} = \frac{2k-r}{2k} \binom{2k}{r}
\]
for all \( r \in \{0, \ldots, 2k\} \), it follows that
\[
\begin{align*}
&\sum_{i=-(2k-1)}^{2k-4} \left( \sum_{j=0}^{2k-3} a_{ij} \frac{(\tau_j)^{2k-r}}{(2k)!} + b_{ij} \frac{(\tau_j)^{2k-1-r}}{(2k-1)!} \right) B_i(\bar{\tau}) \\
&= \left( \frac{2k}{r} \right) \sum_{i=-(2k-1)}^{2k-4} \left( \sum_{j=0}^{2k-3} a_{ij} \frac{(\tau_j)^{2k-r}}{(2k)!} + b_{ij} \frac{(2k-r)(\tau_j)^{2k-1-r}}{(2k)!} \right) B_i(\bar{\tau}) \\
&= \left( \frac{2k}{r} \right) \bar{\tau}^{2k-r} \binom{2k}{r}!
\end{align*}
\]
since for all \( t \in [\tau_0, \tau_{2k-3}] \) and \( 1 \leq r \leq 2k-1 \)
\[
\sum_{i=-(2k-1)}^{2k-4} \left( \sum_{j=0}^{2k-3} a_{ij} (\tau_j)^{2k-r} + b_{ij} (2k-r)(\tau_j)^{2k-1-r} \right) B_i(t) = t^{2k-r}.
\]

Therefore,
\[
\begin{align*}
&= \sum_{i=-(2k-1)}^{2k-4} \left( \sum_{j=0}^{2k-3} a_{ij} \frac{(\tau_j-\bar{\tau})^{2k}}{(2k)!} + b_{ij} \frac{(\tau_j-\bar{\tau})^{2k-1}}{(2k-1)!} \right) B_i(\bar{\tau}) \\
&= \sum_{i=-(2k-1)}^{2k-4} \left( \sum_{j=0}^{2k-3} a_{ij} \frac{(\tau_j)^{2k}}{(2k)!} + b_{ij} \frac{(\tau_j)^{2k-1}}{(2k-1)!} \right) B_i(\bar{\tau})
\end{align*}
\]
\[
\begin{align*}
&+ \left( \sum_{r=1}^{2k} (-1)^r \binom{2k}{r} \right) \bar{\tau}^{2k} (2k)! \\
&= \sum_{i=-(2k-1)}^{2k-4} \left( \sum_{j=0}^{2k-3} a_{ij} \binom{\tau_j}{2k} + b_{ij} \binom{\tau_j}{2k-1} \right) B_i(\bar{\tau}) - \bar{\tau}^{2k} (2k)!
\end{align*}
\]

since \( \sum_{i=-(2k-1)}^{2k-4} \left( \sum_{j=0}^{2k-3} a_{ij} \right) B_i(\bar{\tau}) = 1 \) and \( \sum_{r=0}^{2k} (-1)^r \binom{2k}{r} = 0 \). We conclude that the inequality in (3.17) can be rewritten as stated in the lemma.

3.3. Determining the order of the gap. In this subsection, we show that the gap problem can be reduced to a conjecture concerning the structure of the error bound for a certain Hermite interpolation problem (with uniformity in the knots). We restrict here ourselves to the LSE. In the case of the MLE, the non-linearity of the characterization poses extra difficulties for establishing the stochastic order of the gap. The details of the proof for the MLE are given in the appendix.

The error \( e_k(x) \) defined in Lemma 3.4 can be recognized as a monospline of degree \( 2k \) with \( 2k-2 \) simple knots \( \tau_0, \ldots, \tau_{2k-3} \). For a definition of monosplines, see e.g. Michelli (1972), Bojanov, Hakopian and Sahakian (1993), Nürnberg (1989), page 194 or DeVore and Lorentz (1993), page 136. We will return to this in Lemma 3.8 and the proof of Lemma 3.1 at the end of this section.

Our bound on the random error \( E_n \) will be based on the following conjecture:

**Conjecture 3.1** Let \( 0 = y_0 < y_1 < \cdots < y_{2k-3} = 1 \) be \( 2k-2 \) arbitrary points. If \( g_t(x) = (x - t)^{k-1}/(k-1)! \), then there exists \( c_k > 0 \) depending on \( k \) only such that

\[
\sup_{t \in (0,1)} \sup_{0 < y_1 < \cdots < y_{2k-4} < 1} \| g_t - H g_t \|_\infty \leq c_k.
\]
where $H_g$ denotes the unique interpolating spline of degree $2k-1$ that solves the Hermite problem:

$$H_g(y_j) = g(y_j), \quad (H_g)'(y_j) = g'(y_j)$$

for $j = 0, \ldots, 2k - 3$.

It is shown in Balabdaoui and Wellner (2005) that finiteness of the bound in (3.20) implies that for any function in $C^{(k+j)}, j = 0, \ldots, k$, the supremum norm of the interpolation error $f - Hf$ is also bounded uniformly in the knots $y_1, \ldots, y_{2k-4}$. Proving the above conjecture seems to be hard as little is known about the properties of the spline interpolant. When $f \in C^{(2k)}[0,1]$ such that $\|f^{(2k)}\|_\infty \leq 1$, it is shown in Balabdaoui and Wellner (2005) that the biggest error is achieved pointwise by the so-called perfect spline of degree $2k$ with the same knots as the spline interpolant $Hf$; i.e., for any $t \in [0,1]$

$$\sup_{\|f^{(2k)}\|_\infty \leq 1} |f(t) - [Hf](t)| \leq |S^*(t) - [HS^*](t)|,$$

where

$$S^*(t) = \frac{1}{(2k)!} \left( t^{2k} + 2 \sum_{j=1}^{2k-4} (-1)^j (t - y_j)^{2k} \right)$$

is the perfect spline of degree $2k$ and knots $x_1, \ldots, x_{2k-4}$. In this case, it follows rather easily that

$$\sup_{0<y_1<\cdots<y_{2k-4}<1} \|f - Hf\|_\infty \leq \|f^{(2k)}\|_\infty \sup_{0<y_1<\cdots<y_{2k-4}<1} \|S^* - HS^*\|_\infty.$$

Our numerical investigations suggest strongly that the right hand side of the previous display is bounded, and that it decays at a fast rate as $k$ increases (see Balabdaoui and Wellner (2005), Table 2). Now we will derive an upper bound for the random error $E_n$ based on Conjecture 3.1.
Lemma 3.5 If Conjecture 3.1 holds, then $E_n$ in (3.14) of lemma 3.4 satisfies

$$|E_n| = O_p(n^{-k/(2k+1)}) + o_p((\tau_{2k-3} - \tau_0)^{2k}).$$

Proof. Let $f$ be the function given by

$$f(t) = \sum_{i=-(2k-1)}^{2k-3} \left( \sum_{j=0}^{2k-4} (a_{ij}(\tau_j - t)^{k-1} + b_{ij}(\tau_j - t)^{k-2}) \right) \int_{\tau_j}^{\tau_0} d(G_n(t) - G_0(t)).$$

where $[\tau_j, \bar{\tau}] \equiv [\bar{\tau}, \tau_j]$ if $\tau_j > \bar{\tau}$. Then $E_n$ can be rewritten as

$$E_n = \sum_{i=-(2k-1)}^{2k-3} \left( \sum_{j=0}^{2k-3} (a_{ij}(\tau_j - t)^{k-1} + b_{ij}(\tau_j - t)^{k-2}) \right) \int_{\tau_j}^{\tau_0} d(G_n(t) - G_0(t)).$$

Indeed, we found in the previous subsection that $E_n$ is given by

$$E_n = \sum_{i=-(2k-1)}^{2k-3} \left( \sum_{j=0}^{2k-3} (a_{ij}(\tau_j - t)^{k-1} + b_{ij}(\tau_j - t)^{k-2}) \right) \int_{\tau_j}^{\tau_0} d(G_n(t) - G_0(t)).$$

We write $D_n \equiv Y_n - Y_0$. Then $E_n$ can be rewritten as

$$E_n = \sum_{i=-(2k-1)}^{2k-3} \left( \sum_{j=0}^{2k-3} (a_{ij}(\tau_j - t)^{k-1} + b_{ij}(\tau_j - t)^{k-2}) \right) \int_{\tau_j}^{\tau_0} d(G_n(t) - G_0(t)).$$

Now for arbitrary $x$ and $y$, we can write

$$D_n(y) = D_n(x) + (y - x)D_n'(x) + \cdots + \int_x^y \frac{(y - t)^{k-1}}{(k-1)!} d(G_n(t) - G_0(t))$$

and similarly

$$D_n'(y) = D_n'(x) + (y - x)D_n''(x) + \cdots + \int_x^y \frac{(y - t)^{k-2}}{(k-2)!} d(G_n(t) - G_0(t)).$$

Taking $x = \bar{\tau}$ and $y = \tau_j$ for $j = 0, \ldots, 2k - 3$ and using the identities in (3.19) up to the order $(k - 2)$, it follows that

$$E_n = \sum_{i=-(2k-1)}^{2k-3} \left( \sum_{j=0}^{2k-3} \int_{\tau_j}^{\tau_0} \frac{(\tau_j - t)^{k-1}}{(k-1)!} \right)$$
\[ f(t) = \sum_{i=-(2k-1)}^{2k-4} \left( \sum_{j=0}^{2k-3} \left( a_{ij} \frac{(\tau_j - t)^{k-1}}{(k-1)!} + b_{ij} \frac{(\tau_j - t)^{k-2}}{(k-2)!} \right) + b_{ij} \int_0^t (\tau_j - t)^{k-2} \right) \right) \]

Therefore, the function \( f(t) \) can be viewed as an element of the class of functions

\[ \mathcal{F}_{y_0,R} = \{ f_{y_0,y_1,\ldots,y_{2k-3},\lambda} : y_0 \leq y_1 \leq y_2 \leq \ldots \leq y_{2k-3} \leq y_0 + R, \lambda \in [0,1] \} \]
where \( y_0 \geq x_0 - \delta \) and \( R > 0 \). In view of Conjecture 3.1 together with the triangle inequality,
\[
|f_{y_0,y_{1},...,y_{2k-3},\lambda}(t)| \leq C(y_{2k-3} - y_0)^{k-1}[y_0,y_{2k-3}](t)
\]
and hence the collection \( \mathcal{F}_{y_0,R} \) has envelope function \( F_{y_0,R} \) given by
\[
F_{y_0,R}(t) = CR^{k-1}[y_0,y_0+R](t).
\]
Furthermore, \( \mathcal{F}_{y_0,R} \) is a VC-subgraph collection of functions (see Appendix 6.1 for details of the argument), and hence by van der Vaart and Wellner (1996), Theorem 2.6.7, page 141
\[
sup_Q N(\epsilon \| F \|_{Q,2}, \mathcal{F}_{y_0,R}, L_2(Q)) \leq \left( \frac{K}{\epsilon} \right)^{V_k}
\]
for \( 0 < \epsilon < 1 \) where \( V_k = 2(V(\mathcal{F}) - 1) \) with \( V(\mathcal{F}) \) the VC-dimension of the collection of subgraphs and the constant \( K \) depends only on \( V(\mathcal{F}) \). It follows that
\[
\sup_Q \int_0^1 \sqrt{1 + \log N(\epsilon \| F \|_{Q,2}, \mathcal{F}_{y_0,R}, L_2(Q))} d\epsilon < \infty.
\]
Thus if \( y_0 \in [x_0 - \delta, x_0 + \delta] \) for some small \( \delta > 0 \), then we can find a constant \( M > 0 \) depending only on \( \delta \), and \( g_0 \) such that \( 0 < \sup_{t \in [x_0 - \delta, x_0 + \delta + R]} g_0(t) < M \). Therefore,
\[
EF^2_{y_0,R}(X_1) = C^2 R^{2(k-1)} \int_{y_0}^{y_0+R} g_0(x)dx \leq C^2 MR^{2k-1}.
\]
By van der Vaart and Wellner (1996), Theorem 2.14.1, page 239, it follows that
\[
E \left\{ \sup_{f_{y_0,y_{1},...,y_{2k-3},\lambda} \in \mathcal{F}_{y_0,R}} \left| (G_n - G_0)(f_{y_0,y_{1},...,y_{2k-3},\lambda}) \right| \right\} \leq \frac{K'}{\sqrt{n}} \{ EF^2_{y_0,R}(X_1) \}^{1/2} = O(n^{-1/2}R^{k-1/2})
\]
for some constant \( K' > 0 \) depending only on \( k, x_0, \) and \( \delta \). Application of lemma 5.1 with \( d = k \) and \( \alpha = k \) yields
\[
|E_n| = o_p((\tau_{2k-3} - \tau_0)^{2k}) + O_p(n^{-2k/(2k+1)}).
\]
To show that $\tau_{2k-3} - \tau_0 = O_p(n^{-1/(2k+1)})$, we need the following result:

**Lemma 3.6** The function $x \mapsto e_k(x)$ has no other zeros than $\tau_0, \ldots, \tau_{2k-3}$ in $[\tau_0, \tau_{2k-3}]$.

**Proof.** The result follows from Proposition 1 of Michelli (1972); see also de Boor (2004).

Recall that $e_k(x)$ is a monospline of degree $2k$ with $2k - 2$ simple knots $\tau_0, \ldots, \tau_{2k-3}$. Furthermore, by construction, these knots are also double zeros; i.e. $e_k(\tau_j) = e'_k(\tau_j) = 0$ for $j = 0, \ldots, 2k - 3$. Now, we state two preparatory lemmas that will help determine the sign of the function $e_k(x)$ at any point $x \in \bigcup_{j=0}^{2k-4} (\tau_j, \tau_{j+1})$.

**Lemma 3.7** Let $k \geq 2$ be an integer. The monospline $M_k$ with simple knots $\xi_0 = -k + 3/2, \xi_1 = -k + 5/2, \ldots, \xi_{2k-4} = k - 5/2, \xi_{2k-3} = k - 3/2$ and such that $M_k(\xi_j) = M'_k(\xi_j) = 0$ for $j = 0, \ldots, 2k - 3$ has a constant sign: $-1$ (+1) if $k$ is odd (even).

**Proof.** Consider the function $D_{2k}$ defined on $[-k + 3/2, k - 3/2]$ by

$$D_{2k}(t) = B_{2k}(t - \xi_j) - B_{2k}, \quad \text{on } [\xi_j, \xi_{j+1}] = [\xi_j, \xi_j + 1]$$

for $j = 0, \ldots, 2k - 3$, where $B_{2k}$ is the normalized Bernoulli polynomial of degree $2k$ (defined on $[0, 1]$) and $B_{2k} = B_{2k}(0)$. By the known properties of Bernoulli polynomials (see e.g. Bojanov, Hakopian and Sahakian (1993), pages 117-124), we have $D_{2k}^{(l)}(\xi_j-) = D_{2k}^{(l)}(\xi_j+) = 0$ for $l = 0, \ldots, 2k - 2$. Hence, $D_{2k}$ is a monospline of degree $2k$. Furthermore, since $D_{2k}(\xi_j) = D'_k(\xi_j) = 0$, it follows that $M_k = D_{2k}$ on $[-k + 3/2, k - 3/2]$. Now, the sign of $M_k$ is the same as the sign of $B_{2k} - B_{2k}$ on $[0, 1]$. But latter is determined by the sign
of $B_{2k}(1/2) - B_{2k}$ as 0 and 1 are the only zeros of $B_{2k} - B_{2k}$ on $[0, 1]$. Using the formula

$$B_{2k}(1/2) = -(1 - 2^{1-2k})B_{2k}$$

(see e.g. Abramowitz and Stegun (1972), formula 23.1.21, page 805) and the fact that $B_{2k} > 0$ ($< 0$) when $k$ is odd (even), it follows that $M_k \leq 0$ ($\geq 0$) when $k$ is odd (even).

Lemma 3.8 If $\bar{\tau} \in \cup_{j=0}^{2k-4} (\tau_j, \tau_{j+1})$, then

$$(-1)^ke_k(\bar{\tau}) > 0;$$

i.e., $e_k(\bar{\tau})$ is nonpositive (nonnegative) if $k$ is odd (even).

Proof. Let $\bar{\tau}$ be a fixed point in $\cup_{j=0}^{2k-4} (\tau_j, \tau_{j+1})$. We can assume without loss of generality that $\bar{\tau} \in (\tau_0, \tau_1)$. There exists $\lambda \in (0, 1)$ such that $\bar{\tau} = \lambda \tau_0 + (1 - \lambda)\tau_1$. Consider now the function

$$(\tau_0, \ldots, \tau_{2k-3}) \mapsto \frac{e_k(\bar{\tau}) + |e_k(\bar{\tau})|}{2e_k(\bar{\tau})}.$$ 

Note that it is possible to divide by $e_k(\bar{\tau})$ since $e_k(\bar{\tau}) \neq 0$ as $\bar{\tau}$ is different from the knots. It is easy to see that the function is continuous in $\tau_0, \ldots, \tau_{2k-3}$. Furthermore, it can only take two possible values, 0 or 1, and therefore has to be constant. But, when the knots are equally distant, we know from Lemma 3.7 that the constant is 0 (1) if $k$ is odd (even). It follows that $(-1)^ke_k(\bar{\tau}) > 0$.

Proof of Lemma 3.1. Let $j_0 \in \{0, \ldots, 2k-4\}$ be such that $[\tau_{j_0}, \tau_{j_0+1}]$ is the largest knot interval; i.e., $\tau_{j_0+1} - \tau_{j_0} = \max_{0 \leq j \leq 2k-4}(\tau_{j+1} - \tau_j)$. Let $a = \tau_0$, $b = \tau_{2k-3}$. By Conjecture 3.1, there exists a constant $D > 0$ depending only on $k$ such that

$$|R_n| \leq D \sup_{t \in [\tau_0, \tau_{2k-3}]} |g_0^{(k)}(t) - g_0^{(k)}(\bar{\tau})| (\tau_{2k-3} - \tau_0)^{2k}.$$
In the previous bound, we used the fact that the $2k$-times derivative of the function

$$x \mapsto \int_{\bar{\tau}}^{x} \frac{(x - t)^{2k-1}}{(2k-1)!} (g_0^{(k)}(t) - g_0^{(k)}(\bar{\tau}))dt$$

is $g_0^{(k)}(x) - g_0^{(k)}(\bar{\tau})$. By uniform continuity of $g_0^{(k)}$ on compacts, it follows that

$$|\mathbb{R}_n| = o_p((\tau_{2k-3} - \tau_0)^{2k}).$$

Using the result of Lemma 3.4 and since the bounds on $\mathbb{R}_n$ and $E_n$ (see Lemma 3.5) are independent of the choice of $\bar{\tau}$ in $\bigcup_{j=0}^{2k-4} (\tau_j, \tau_{j+1})$, it follows that

$$\sup_{\bar{\tau} \in (\tau_{j0}, \tau_{j0+1})} (-1)^k e_k(\bar{\tau}) \leq O_p(n^{-2k/(2k+1)} + o_p((\tau_{2k-3} - \tau_0)^{2k}).$$

Now, on the interval $[\tau_{j0}, \tau_{j0+1}]$, the Hermite interpolation part of the error $e_k$ is a polynomial of degree $2k - 1$. On the other hand, the best uniform approximation of the function $x^{2k}$ on $[\tau_{j0}, \tau_{j0+1}]$ from the space of polynomials of degree $\leq 2k - 1$ is given by the polynomial

$$(3.24) x \mapsto x^{2k} - \left(\frac{\tau_{j0+1} - \tau_{j0}}{2}\right)^{2k} \frac{1}{2^{2k-1}(2k)!} T_{2k} \left(\frac{2x - (\tau_{j0} + \tau_{j0+1})}{\tau_{j0+1} - \tau_{j0}}\right),$$

where $T_{2k}$ is the Chebyshev polynomial of degree $2k$ (defined on $[-1, 1]$), see, e.g., NÜRNBERGER (1989), Theorem 3.23, page 46 or DeVORE and LORENTZ (1993), Theorem 6.1, page 75. It follows that

$$(3.25) \sup_{\bar{\tau} \in (\tau_{j0}, \tau_{j0+1})} (-1)^k e_k(\bar{\tau}) \geq \left\| \frac{T_{2k}}{2^{4k-1}(2k)!} (\tau_{j0+1} - \tau_{j0})^{2k} \right\|_{\infty} = \frac{1}{2^{4k-1}(2k)!} (\tau_{j0+1} - \tau_{j0})^{2k}$$

since $\|T_{2k}\|_{\infty} = 1$. But,

$$\tau_{2k-3} - \tau_0 = \sum_{j=0}^{2k-4} (\tau_{j+1} - \tau_j) \leq (2k - 3)(\tau_{j0+1} - \tau_{j0})$$
It follows that
\[
\sup_{\bar{\tau} \in (\tau_j, \tau_{j+1})} (-1)^k e_k(\bar{\tau}) \geq \frac{1}{(2k-3)2k2^{4k-1}(2k)!}\left((\tau_{2k-3} - \tau_0)^{2k}\right).
\]

Combining the results obtained above, we conclude that
\[
\frac{(-1)^k g_0^{(k)}(x_0)}{(2k-3)2k2^{4k-1}(2k)!}\left((\tau_{2k-3} - \tau_0)^{2k}\right) \leq O_p\left(n^{-2k/(2k+1)}\right) + o_p\left((\tau_{2k-3} - \tau_0)^{2k}\right)
\]
which implies that \(\tau_{2k-3} - \tau_0 = O_p(n^{-1/(2k+1)})\). ●

4. Proofs for Subsection 3.1. To prove Theorem 2.2 we first need to establish the key rate Proposition 2.1, as it will be used to derive the rates of convergence of \(\bar{g}_n^{(j)}\), \(j = 0, \ldots, k - 1\) at a fixed point \(x_0\).

Proving the key rate Proposition 2.1 relies crucially on the following “existence of point” proposition. Consider the event \(J_n = J_n^{(1)} \cap J_n^{(2)}\) where \(J_n^{(i)}\), \(i = 1, 2\), are defined by

\[
J_n^{(1)} = J_n^{(1)}(x_0, k, M) = \{\text{there exist (}k + 1\text{) jump points } \tau_{n,1}, \ldots, \tau_{n,k+1} \text{ (not necessarily successive) satisfying} \}
\]
\[
x_0 - n^{-1/(2k+1)} \leq \tau_{n,1} < \cdots < \tau_{n,k+1} \leq x_0 + M n^{-1/(2k+1)}
\]
\[
k n^{-1/(2k+1)} \leq \tau_{n,k+1} - \tau_{n,1} \leq M n^{-1/(2k+1)} \}
\]

and

\[
J_n^{(2)} = J_n^{(2)}(j, k, c_j) = \left\{\inf_{t \in [\tau_{n,1}, \tau_{n,k+1}]} |\hat{g}_n^{(j)}(t) - g_0^{(j)}(t)| \leq c_j n^{-(k-j)/(2k+1)}\right\}.
\]

Proposition 4.1 Suppose that \((-1)^k g_0^{(k)}(x_0) > 0\) and \(g_0^{(k)}\) is continuous in a neighborhood of \(x_0\). Let \(\hat{g}_n\) be either the MLE \(\hat{g}_n\) or the LSE \(\tilde{g}_n\) and let \(0 \leq j \leq k - 1\). Suppose also that \(\int_0^\infty y^{-1/2}dG_0(y) < \infty\) holds. Then, if Conjecture 3.1 holds, for any \(\epsilon > 0\), there exists \(M > 0\) and \(c_j > 0\) such that \(P(J_n) > 1 - \epsilon\) for all sufficiently large \(n\).
Proof of Proposition 4.1. Fix $\epsilon > 0$. In what follows, we consider only the LSE since the result in the case of the MLE can be proved similarly by using the same perturbation functions and uniform consistency of the estimator. We will start with $j = 0$. Fix $\epsilon > 0$. For ease of notation, we will write the jump points of $\tilde{g}_n^{(k-1)}$ without the subscript $n$. Let $\tau_1$ be the first jump point of $\tilde{g}_n^{(k-1)}$ after $x_0 - n^{-1/(2k+1)}$, $\tau_2$ the first jump point after $\tau_1 + n^{-1/(2k+1)}$, \ldots, $\tau_{k+1}$ the first jump point after $\tau_k + n^{-1/(2k+1)}$. By Lemma 2.1, there exists $M > 0$ such that

$$0 \leq \tau_{k+1} - \tau_1 \leq M n^{-1/(2k+1)}$$

with probability $> 1 - \epsilon$. Note that by construction $\tau_{k+1} - \tau_1 \geq k n^{-1/(2k+1)}$. Fix $c > 0$ and consider the event

$$\inf_{t \in [\tau_1, \tau_{k+1}]} |\tilde{g}_n(t) - g_0(t)| > cn^{-k/(2k+1)}. \tag{4.26}$$

On this set and for any nonnegative function $g$ on $[\tau_1, \tau_{k+1}]$, we have

$$\int_{\tau_1}^{\tau_{k+1}} (\tilde{g}_n(t) - g_0(t)) g(t) dt \geq cn^{-k/(2k+1)} \int_{\tau_1}^{\tau_{k+1}} g(t) dt. \tag{4.27}$$

Now, let $B$ be the B-spline of degree $k-1$ and with support $[x_1, x_{k+1}]$. Recall from (3.11) in Section 5 that $B$ can be given by

$$B(t) = [\tau_1, \ldots, \tau_{k+1}]k (-t)^{k-1}$$

where $[x_1, \ldots, x_m]g$ denotes the divided difference of degree $m$ with respect to the points $x_1, \ldots, x_m$. The B-spline $B$ can be given more explicitly by

$$B(t) = (-1)^k k \left( \frac{(t - \tau_1)^{k-1}}{\prod_{j \neq 1}(\tau_j - \tau_1)} + \cdots + \frac{(t - \tau_k)^{k-1}}{\prod_{j \neq k}(\tau_j - \tau_k)} \right).$$

for all $t \in [\tau_1, \tau_{k+1}]$. Let $|\eta| > 0$ and consider the perturbation function

$$p(t) = \prod_{1 \leq i < j \leq k+1} (\tau_j - \tau_i) \times B(t).$$

It is easy to check that for $|\eta|$ small enough, the perturbed function

$$\tilde{g}_{n,\eta}(t) = \tilde{g}_n(t) + \eta p(t)$$

is...
is \(k\)-monotone on \((0, \infty)\). Indeed, \(p\) was chosen so that it satisfies \(p^{(j)}(\tau_1) = p^{(j)}(\tau_{k+1}) = 0\) for \(0 \leq j \leq k - 2\), which guarantees that the perturbed function \(\tilde{g}_{\eta,n}\) belongs to \(C^{k-2}(0, \infty)\). For \(0 \leq j \leq k - 3\), the properties of strict convexity and monotonicity of \((-1)^j \tilde{g}^{(j)}_n\) on \((0, \infty)\) are preserved by \(\tilde{g}^{(j)}_{\eta,n}\) as long as \(|\eta|\) is small enough. For \(k - 2\), \((-1)^{k-2} \tilde{g}^{(k-2)}_{\eta,n}\) is a convex and nonincreasing on \((0, \infty)\) piecewise linear function. Now note that \(p\) is a spline of degree \(k - 1\) whose knots are included in the set of knots of \(\tilde{g}_n\). Moreover, for small values of \(|\eta|\) it can be easily checked that \((-1)^{k-2} \tilde{g}^{(k-2)}_{\eta,n}\) is nonincreasing and convex on \((0, \infty)\). It follows that

\[
\lim_{\eta \to 0} \frac{Q_n(\tilde{g}_{\eta,n}) - Q_n(\tilde{g}_n)}{\eta} = 0.
\]

This implies that

\[
\int_{\tau_1}^{\tau_{k+1}} p(t) d(\tilde{G}_n - G_n)(t) = 0.
\]

The previous equality can be rewritten as

\[
\int_{\tau_1}^{\tau_{k+1}} p(t) (\tilde{g}_n(t) - g_0(t)) dt = \int_{\tau_1}^{\tau_{k+1}} p(t) d(G_n(t) - G_0(t)).
\]

Taking \(g \equiv p\) in (4.27), we obtain

\[
\left| \int_{\tau_1}^{\tau_{k+1}} p(t) d(G_n(t) - G_0(t)) \right| \geq cn^{-k/(2k+1)} \int_{\tau_1}^{\tau_{k+1}} p(t) dt \geq cn^{-k/(2k+1)} \prod_{1 \leq i < j \leq k+1} (\tau_j - \tau_i)
\]

(4.28)

\[
\geq cn^{-k/(2k+1)} \left( \frac{n-1}{2k+1} \right)^{k(k+1)/2}
\]

(4.29)

where in (4.28), we used the fact that B-splines integrate to 1, whereas in (4.29) we used the facts that there are \(k(k + 1)/2\) terms in the product \(\prod_{1 \leq i < j \leq k+1} (\tau_j - \tau_i)\) and that \(\tau_j - \tau_i \geq n^{-1/(2k+1)}\), \(1 \leq i < j \leq k + 1\). Let \(0 < y_0 < y_1 < \cdots < y_{k-1} < y_k\) be \((k + 1)\) points in \((0, \infty)\) and consider the function \(f_{y_0, y_1, \ldots, y_{k-1}, y_k}\) defined by

\[
f_{y_0, \ldots, y_k}(t) = (-1)^k k \prod_{0 \leq i < j \leq k} (y_j - y_i) \sum_{l=0}^{k-1} \frac{(y_l - t)^{k-1}}{\prod_{j \neq l} (y_j - y_l)}
\]
where

\[
\alpha_j = (-1)^k k \prod_{0 \leq l < l' \leq k} y_{l'} - y_l \prod_{j' \neq j} y_{j'} - y_j.
\]

Let \( R > 0 \) and consider the collection of functions

\[
F_{y_0,R} = \{ f_{y_0,y_1,\ldots,y_{k-1},y_k} : y_0 < y_1 < \cdots < y_{k-1} < y_k \leq y_0 + R \}.
\]

We first find an envelope function for the class \( F_{y_0,R} \). Note that for \( j = 0, \ldots, k \), the product \( \prod_{j' \neq j} (y_{j'} - y_j) \) contains \( k \) terms and hence \( \alpha_j \) is a product of \( k(k+1)/2 - k = k(k-1)/2 \) terms that are at most \( R \) distant from one another. It follows that

\[
\alpha_j \leq kR^{k(k-1)/2}, \quad \text{for } j = 0, \ldots, k.
\]

Thus the functions being summed in (4.30) have common envelope \( kR^{k(k-1)/2}(y_0 + R - t)_{+}^{k-1}[y_0,y_0+R](t) \), and this yields the envelope

\[
F_{y_0,R}(t) = k^2R^{k(k-1)/2}(y_0 + R - t)_{+}^{k-1}[y_0,y_0+R](t)
\]

for the class \( F_{y_0,R} \). Furthermore, \( F_{y_0,R} \) is a VC-subgraph collection of functions (see Appendix 6.1 for details of the argument), and hence by van der Vaart and Wellner (1996), Theorem 2.6.7, page 141,

\[
\sup_Q N \left( \epsilon \| F_{y_0,R} \|_{Q,2}, F_{y_0,R}, L_2(Q) \right) \leq \left( \frac{K}{\epsilon} \right)^{V_k}.
\]

for \( 0 < \epsilon < 1 \) where \( V_k = 2(V(F) - 1) \) with \( V(F) = V(F_{y_0,R}) \) the VC-dimension of the collection of subgraphs. Therefore

\[
\sup_Q \int_0^1 \sqrt{1 + \log(N \left( \epsilon \| F_{y_0,R} \|_{Q,2}, F_{y_0,R}, L_2(Q) \right))} \, d\epsilon < \infty.
\]

On the other hand, if \( y_0 \) is in a small neighborhood \([x_0 - \delta, x_0 + \delta] \) for some small \( \delta > 0 \), there exists some constant \( C > 0 \) depending only on \( \delta, R \) and
$g_0(x_0)$ such that $0 < g_0 < C$ on $[y_0, y_0 + R]$ for all $y_0 \in [x_0 - \delta, x_0 + \delta]$. It follows that

$$
E F_{y_0, R}^2(X_1) \leq k^4 R^{k(k-1)} \int_{y_0}^{y_0+R} (y_0 + R - x)^{2k-2} g_0(x) dx
$$

$$
\leq \frac{k^4 C}{2k-1} R^{k(k-1)} R^{2k-1} = \frac{k^4 C}{2k-1} R^{k(k+1)-1}.
$$

Therefore, by van der Vaart and Wellner (1996), Theorem 2.14.1, we have

$$
E \left\{ \left( \sup_{f_{y_0,y_1,\ldots,y_k} \in F_{y_0,R}} \left| (G_n - G_0)(f_{y_0,y_1,\ldots,y_k}) \right| \right)^2 \right\}
\leq \frac{K'}{n} E F_{y_0, R}^2(X_1) = O(n^{-1} R^{k(k+1)-1}),
$$

(4.31)

for some constant $K'$ depending only on $k$, $x_0$, and $\delta$. Application of lemma 5.1 with $d = k(k + 1)/2$ and $\alpha = k$ yields

$$
\left| (P_n - P_0)(f_{y_0,y_1,\ldots,y_k}) \right| \leq \epsilon (y_k - y_0)^{(3+k)k/2} + O_p \left( n^{-((3+k)k/(2(2k+1)))} \right)
$$

uniformly in $y_0, \ldots, y_k$. It follows that

$$
\left| \int_{\tau_1}^{\tau_{k+1}} p(t) d(G_n - G_0)(t) \right| = O_p \left( n^{-((3+k)k/(2(2k+1)))} \right)
$$

and we can choose $c_0 = c$ to be large enough so that the probability of the event (4.26) is arbitrarily small. This proves the result for $j = 0$. Now let $1 \leq j \leq k - 1$. This time we will need $(k + 1 + j)$ jump points $\tau_1 < \cdots < \tau_{k+1+j}$. As for $j = 0$, $\tau_1$ is taken to be the first jump point of $g_n^{(k-1)}$ after $x_0 - n^{-1/(2k+1)}$, $\tau_2$ the first jump point after $\tau_1 + n^{-1/(2k+1)}$ and so on. Notice that the existence of at least $k + 1 + j$ jump points is guaranteed by the fact that $g_n^{(k)}(x_0) \neq 0$ which implies that with probability 1, the number of jump points tends to infinity with increasing sample size $n$. Consider the function

$$
g_j(t) = \prod_{1 \leq i < j \leq k+j+1} (\tau_j - \tau_i) \times B_j(t)
$$
where $B_j$ is the B-spline of degree $k + j - 1$ with support $[\tau_1, \tau_{k+1+j}]$; i.e.,

$$B_j(t) = (-1)^{k+j}(k+j) \left( \frac{(\tau_1 - t)^{k+j-1}}{\prod_{j \neq 1}(\tau_j - \tau_1)} + \cdots + \frac{(\tau_{k+j} - t)^{k+j-1}}{\prod_{j \neq k+j}(\tau_j - \tau_{k+j})} \right).$$

It is easy to check that $p_j = q_j^{(j)}$ is a valid perturbation function (it is a spline of degree $k - 1$) since for $|\eta|$ small enough, the function

$$\tilde{g}_{\eta,n,j} = \tilde{g}_n + \eta p_j$$

is $k$-monotone. It follows that

$$\lim_{\eta \to 0} \frac{Q_n(\tilde{g}_{\eta,n,j}) - Q_n(\tilde{g}_n)}{\eta} = 0$$

which implies that

$$\int_{\tau_1}^{\tau_{k+1+j}} p_j(t) (\tilde{g}_n(t) - g_0(t)) dt = \int_{\tau_1}^{\tau_{k+1+j}} p_j(t) d(G_n(t) - G_0(t)) dt$$

By successive integrations by parts and using the fact that $q_j^{(i)}(\tau_1) = q_j^{(i)}(\tau_{k+1+j}) = 0$ for $i = 0, \ldots, k+j-2$, we obtain

$$\int_{\tau_1}^{\tau_{k+1+j}} (-1)^{j} q_j(t) (\tilde{g}_n^{(j)}(t) - g_0^{(j)}(t)) dt = \int_{\tau_1}^{\tau_{k+1+j}} p_j(t) d(G_n(t) - G_0(t)) dt.$$

Therefore, if we assume that there exists $c > 0$ such that

$$\inf_{t \in [\tau_1, \tau_{k+1+j}]} \left| \tilde{g}_n^{(j)}(t) - g_0^{(j)}(t) \right| > c n^{-(k-j)/(2k+1)}$$

then

$$\left| \int_{\tau_1}^{\tau_{k+1+j}} p_j(t) d(G_n(t) - G_0(t)) dt \right| \geq c n^{-(k-j)/(2k+1)} \int_{\tau_1}^{\tau_{k+1+j}} q_j(t) dt$$

$$\geq c (k+j) n^{-(k-j)/(2k+1)} \left( n^{-1/(2k+1)} \right)^{(k+1+j)(k+2+j)/2}$$

$$= c (k+j) n^{-(2(k-j)+(k+j)(k+j+1))/(2(2k+1))}$$

$$= c (k+j) n^{-(3k-j+(k+j)^2)/(2(2k+1))}.$$
Using similar empirical process arguments as in the proof for \( j = 0 \) together with an application of lemma 5.1 with \( 2d = 3k - j + (k + j)^2 \) and \( \alpha = k \), it follows that

\[
\left| \int_{\tau_n}^{\tau_n t_k + 1 + j} p_j(t) d\left( G_n(t) - G_0(t) \right) dt \right| = O_p \left( n^{-\frac{(3k-j+(k+j)^2)}{2(2k+1)}} \right)
\]

and the result for \( 1 \leq j \leq k - 1 \) follows.

With Proposition 4.1 in hand we are prepared for the proof of the rates Proposition 2.1.

**Proof of Proposition 2.1.** To prove (2.2), we will use induction starting from the highest order of differentiation \( k - 1 \). The techniques used here are very much analogous to the ones used in the case \( k = 2 \) in Groeneboom, Jongbloed, and Wellner (2001b). But this was possible mainly because of the result established in the previous lemma. We begin by establishing (2.1).

Let \( M > 0 \) and \( 0 < \epsilon < 1 \). We consider two sequences of \((k+1)\) jump points \( \tau_{1,1}, \ldots, \tau_{k+1,1} \) and \( \tau_{1,2}, \ldots, \tau_{k+1,2} \) as described in the previous proposition, where \( \tau_{1,1} \) is the first jump point of \( g_n^{(k-1)} \) after \( x_0 + Mn^{-1/(2k+1)} \) and \( \tau_{1,2} \) is the first jump after \( \tau_{k+1,1} + n^{-1/(2k+1)} \). Similarly, we define two other sequences \( \tau_{1,-1}, \ldots, \tau_{k+1,-1} \) and \( \tau_{1,-2}, \ldots, \tau_{k+1,-2} \) to the left of \( x_0 \). By the previous theorem, we can find \( c > 0 \) so that,

\[
\inf_{t \in [\tau_{1,i}, \tau_{k+1,i}]} \left| g_n^{(k-2)}(t) - g_0^{(k-2)}(t) \right| < cn^{-2/(2k+1)}
\]

for \( i = -2, -1, 1, 2 \) with probability greater than \( 1 - \epsilon \). Let \( \xi_1 \) and \( \xi_2 \) be the minimizer of \( |g_n^{(k-2)} - g_0^{(k-2)}| \) on \([\tau_{1,1}, \tau_{k+1,1}]\) and \([\tau_{1,2}, \tau_{k+1,2}]\) respectively. Define \( \xi_{-1} \) and \( \xi_{-2} \) similarly to the left of \( x_0 \). For all \( t \in [x_0 - Mn^{-1/(2k+1)}, x_0 + Mn^{-1/(2k+1)}] \), we have with probability greater than \( 1 - \epsilon \)

\[
(-1)^{k-2} g_n^{(k-1)}(t^-) \leq (-1)^{k-2} g_n^{(k-1)}(t^+) \leq \frac{(-1)^{k-2} g_0^{(k-2)}(\xi_2) - (-1)^{k-2} g_n^{(k-2)}(\xi_1)}{\xi_2 - \xi_1} \leq \frac{(-1)^{k-2} g_0^{(k-2)}(\xi_2) - (-1)^{k-2} g_0^{(k-2)}(\xi_1) + 2cn^{-2/(2k+1)}}{\xi_2 - \xi_1} \leq (-1)^{k-2} g_0^{(k-1)}(\xi_2) + 2cn^{-1/(2k+1)}
\]
since $\xi_2 - \xi_1 \geq n^{-1/(2k+1)}$. Similarly, with probability greater than $1 - \epsilon$, we have that

$$( -1 )^{k-2} \frac{g_n^{(k-1)}(t+)}{g_n^{(k-1)}(t-)} \geq \frac{( -1 )^{k-2} g_n^{(k-1)}(t-)}{g_n^{(k-1)}(t-)} \geq ( -1 )^{k-2} g_0^{(k-1)}(\xi_2 - 2cn^{-1/(2k+1)}).
$$

Now, using the fact that $\xi_{2} = x_0 + O_p(n^{-1/(2k+1)})$ and differentiability of $g_0^{(k-1)}$ at the point $x_0$, we obtain (2.1). Using similar arguments in the proof of Lemma 4.4 in Groeneboom, Jongbloed, and Wellner (2001b), we can show (2.2) for $j = k - 2$ which specializes to

$$
\sup_{|t| \leq M} \left| \frac{g_n^{(k-2)}(x_0 + n^{-1/(2k+1)}t)}{n^{-1/(2k+1)}t} - g_0^{(k-2)}(x_0) \right| = O_p(n^{-2/(2k+1)})
$$

for all $M > 0$. Indeed, since the jump points $\tau_{j,i}$, $j = 1, \ldots, k+1, i = -2, -1, 1, 2$ are at distance from $x_0$ that is $O_p(n^{-1/(2k+1)})$, we can find with probability exceeding $1 - \epsilon$, $K > M$ such that $\xi_1$ and $\xi_2$ are in $[x_0 - Kn^{-1/(2k+1)}, x_0 + Kn^{-1/(2k+1)}]$, $\xi_2$ and $\xi_1$ in $[x_0 - Kn^{-1/(2k+1)}, x_0 + Mn^{-1/(2k+1)}]$. But we know that, with probability greater than $1 - \epsilon$, we can find $c > 0$ such that

$$
|g_n^{(k-2)}(\xi_1) - g_0^{(k-2)}(\xi_1)| \leq cn^{-2/(2k+1)}.
$$

Also, with probability greater than $1 - \epsilon$, we can find $c' > 0$ such that

$$
\sup_{t \in [x_0 - Kn^{-1/(2k+1)}, x_0 + Kn^{-1/(2k+1)}]} \left| \frac{g_n^{(k-1)}(t) - g_0^{(k-1)}(x_0)}{t} \right| \leq c' n^{-1/(2k+1)}.
$$

Hence, with probability greater than $1 - 3\epsilon$, we have for any $t \in [x_0 - Mn^{-1/(2k+1)}, x_0 + Mn^{-1/(2k+1)}]$

$$
( -1 )^{k-2} \frac{g_n^{(k-2)}(t)}{g_0^{(k-2)}(x_0)} \geq ( -1 )^{k-2} \frac{g_n^{(k-2)}(\xi_1)}{g_n^{(k-1)}(t-)} + ( -1 )^{k-2} g_n^{(k-1)}(\xi_1)(t - \xi_1)
$$

$$
\geq ( -1 )^{k-2} g_0^{(k-2)}(\xi_1) - cn^{-2/(2k+1)} + ( -1 )^{k-2} g_0^{(k-1)}(x_0)
$$

$$
+ c' n^{-1/(2k+1)}(t - \xi_1)
$$

$$
\geq ( -1 )^{k-2} g_0^{(k-2)}(x_0) + (\xi_1 - x_0)( -1 )^{k-2} g_0^{(k-1)}(x_0)
$$
\[
\nu_j \text{ holds for } \mathcal{K}_n
\]

be taken to be the largest neighborhood where \( g_n \) is taken sufficiently large so that \([x_0 - \delta, x_0 + \delta] \subseteq [x_0 - \delta, x_0 + \delta]\). We conclude that (2.2) holds for \( j = k - 2 \). Now, suppose that (2.2) is true for all \( j' > j - 1 \); i.e.,

where in (4.33), we used convexity of \(-1)k-2g_0^{(k-2)} \) “from below”. On the other hand, using convexity of \(-1)k-2g_0^{(k-2)} \) but this time “from above”, we have

\[
\begin{align*}
&(-1)k-2g_0^{(k-2)}(t) \\
&\leq (-1)k-2g_0^{(k-2)}(t) - cn^{-2/(2k+1)} - c'n^{-1/(2k+1)}(\xi_1 - t) \\
&\leq (-1)k-2g_0^{(k-2)}(t) - cn^{-2/(2k+1)} \\
&\quad + \frac{1}{2}(\xi_1 - x_0)^2(-1)k-2g_0^{(k)}(\nu) \\
&\quad + (-1)k-2g_0^{(k-1)}(t - \xi_1) + 2cn^{-2/(2k+1)} \frac{t - \xi_1}{\xi_1 - \xi_1} \\
&\leq (-1)k-2g_0^{(k-2)}(t) + (\xi_1 - x_0)(-1)k-2g_0^{(k-2)}(x_0) \\
&\quad + \frac{1}{2}(\xi_1 - x_0)^2(-1)k-2g_0^{(k)}(\nu) \\
&\quad + \left((-1)k-2g_0^{(k-1)}(x_0) + c'n^{-1/(2k+1)}\right) (t - \xi_1) + 2cn^{-2/(2k+1)} \frac{t - \xi_1}{\xi_1 - \xi_1} \\
&\leq (-1)k-2g_0^{(k-2)}(x_0) + (t - x_0)(-1)k-2g_0^{(k-1)}(x_0) \\
&\quad + \left(D_1 + 2c + 2Kc'\right) n^{-2/(2k+1)}
\end{align*}
\]

where \( \nu \in (\xi_1, x_0) \), \( D_1 = \sup_{x \in [x_0 - \delta, x_0 + \delta]} |g_0^{(k)}(x)| \) and \([x_0 - \delta, x_0 + \delta] \) can be taken to be the largest neighborhood where \( g_0^{(k)} \) exists and is continuous.

In all the previous calculations, \( n \) is taken sufficiently large so that \([x_0 - Kn^{-1/(2k+1)}, x_0 + Kn^{-1/(2k+1)}] \subseteq [x_0 - \delta, x_0 + \delta]\). We conclude that (2.2) holds for \( j = k - 2 \). Now, suppose that (2.2) is true for all \( j' > j - 1 \); i.e.,
for all $M > 0$

$$\sup_{|t| < M} \left| \bar{g}_n^{(j')}(x_0 + n^{-1/(2k+1)}t) - \sum_{i=j'}^{k-1} \frac{n^{-(i-j')/(2k+1)} g_0^{(i)}(x_0) t^{i-j'}}{(i-j')!} \right| = O_p(n^{-(k-j')/(2k+1)}).$$

We are going to prove (2.2) for $j - 1$. We assume without loss of generality that $k$ and $j - 1$ are even. In what follows, $\xi_{\pm 1}$ denotes the same numbers introduced before but this time there are associated with $\bar{g}_n^{(j-1)}$; i.e., for any $0 < \epsilon < 1$, there exist $c > 0$ and $K > M$ such that

$$|\bar{g}_n^{(j-1)}(\xi_{\pm 1}) - g_0^{(j-1)}(\xi_{\pm 1})| \leq cn^{-(k-j+1)/(2k+1)}$$

with probability greater than $1 - \epsilon$ and where $\xi_1 \in [x_0 + Mn^{-1/(2k+1)}, x_0 + Kn^{-1/(2k+1)}]$ and $\xi_{-1} \in [x_0 - Kn^{-1/(2k+1)}, x_0 - Mn^{-1/(2k+1)}]$. Now, using the induction assumption, we know that we can find $c' > 0$ such that, with probability greater than $1 - \epsilon$,

$$-c'n^{-(k-j')/(2k+1)} \leq \bar{g}_n^{(j')}(x_0 + n^{-1/(2k+1)}t) - \sum_{i=j'}^{k-1} \frac{n^{-(i-j')/(2k+1)} g_0^{(i)}(x_0) t^{i-j'}}{(i-j')!} \leq c'n^{-(k-j')/(2k+1)}$$

(4.34)

for all $|t| \leq M$ and $j' > j - 1$. Using convexity of $\bar{g}_n^{(j-1)}$ “from below”, we have for all $|t - x_0| \leq Mn^{-1/(2k+1)}$ with probability greater than $1 - 2\epsilon$, $\bar{g}_n^{(j-1)}(t)$

$$\geq \bar{g}_n^{(j-1)}(\xi_1) + \bar{g}_n^{(j)}(\xi_1)(t - \xi_1) + \cdots + \frac{1}{(k-j)!} \bar{g}_n^{(k-1)}(\xi_1)(t - \xi_1)^{k-j}$$

$$\geq g_0^{(j-1)}(\xi_1) - cn^{-(k-j+1)/(2k+1)} + \left( \sum_{i=j}^{k-1} \frac{g_0^{(i)}(x_0)}{(i-j)!} (\xi_1 - x_0)^{i-j} (t - \xi_1) \right)$$

$$+ \left( \sum_{i=j+1}^{k-1} \frac{g_0^{(i)}(x_0)}{(i-j-1)!} (\xi_1 - x_0)^{i-j-1} \right) \frac{(t - \xi_1)^2}{2!}$$

$$+ \cdots + g_0^{(k-1)}(x_0) \frac{(t - \xi_1)^{k-j}}{(k-j)!}$$
where the right side of (4.35) can be bounded below by

\( g \cdot \sum_{k=0}^{\infty} \frac{g_0^{(k)}(x_0)}{(k-j)!} (\xi_1 - x_0)^{k-j+1} \)

Using Taylor expansion of \( g_0^{(j-1)}(x_1) \) around \( g_0^{(j-1)}(x_0) \), we can write

\[
\begin{align*}
g_0^{(j-1)}(\xi_1) &= g_0^{(j-1)}(x_0) + g_0^{(j)}(x_0)(\xi_1 - x_0) + \cdots + g_0^{(k-1)}(x_0)(\xi_1 - x_0)^{k-j} + \frac{g_0^{(k)}(x_0)}{(k-j)!}(\xi_1 - x_0)^{k-j+1} \\

&= g_0^{(j-1)}(x_0) + g_0^{(j)}(x_0)(t - x_0) + \frac{g_0^{(k-1)}(x_0)}{(k-j)!}(\xi_1 - x_0)^{k-j+1} \\

&\quad + \frac{g_0^{(k)}(x_0)}{(k-j+1)!}(\xi_1 - x_0)^{k-j+1}
\end{align*}
\]

where \( \nu \in (x_0, \xi_1) \). Using this expansion and the fact that \( |t - \xi_1| \leq Kn^{-1/(2k+1)} \), the right side of (4.35) can be bounded below by

\[
\begin{align*}
\sum_{i=j}^{\infty} \frac{g_0^{(i)}(x_0)}{(i-j+1)!} (\xi_1 - x_0)^{i-j+1} + \sum_{i=j}^{\infty} \frac{g_0^{(i)}(x_0)}{(i-j)!} (\xi_1 - x_0)^{i-j}(t - \xi_1) \\
+ \sum_{i=j+1}^{\infty} \frac{g_0^{(i)}(x_0)}{(i-j-1)!} (\xi_1 - x_0)^{i-j-1}(t - \xi_1)^2 + \cdots + \frac{g_0^{(k-1)}(x_0)}{(k-j)!} (\xi_1 - x_0)^{k-j} \\
- \left( c + c' \sum_{p=1}^{k-j} \frac{K^p}{p!} \right) n^{-(k-j+1)/(2k+1)} + \frac{g_0^{(k)}(\nu)}{(k-j+1)!}(\xi_1 - x_0)^{k-j+1} \\
= g_0^{(j-1)}(x_0) + g_0^{(j)}(x_0)(t - x_0) \\
+ \frac{g_0^{(j+1)}(x_0)}{2!} ((\xi_1 - x_0)^2 + 2(\xi_1 - x_0)(t - \xi_1) + (t - \xi_1)^2) \\
+ \cdots + \frac{g_0^{(k-1)}(x_0)}{(k-j)!} \sum_{p=0}^{k-j} \frac{(k-j)!}{(k-j-p)!p!} (\xi_1 - x_0)^{k-j-p} (t - \xi_1)^p \\
- \left( c + c' \sum_{p=1}^{k-j} \frac{K^p}{p!} \right) n^{-(k-j+1)/(2k+1)} + \frac{g_0^{(k)}(\nu)}{(k-j+1)!}(\xi_1 - x_0)^{k-j+1} \\
= g_0^{(j-1)}(x_0) + g_0^{(j)}(x_0)(t - x_0) + \cdots + \frac{g_0^{(k-1)}(x_0)}{(k-j)!} (t - x_0)^{k-j} \\
- \left( c + c' \sum_{p=1}^{k-j} \frac{K^p}{p!} \right) n^{-(k-j+1)/(2k+1)} - \frac{D_1 K^{k-j+1}}{(k-j+1)!} n^{-(k-j+1)/(2k+1)}
\end{align*}
\]

since \( 0 \leq \xi_1 - x_0 \leq Kn^{-1/(2k+1)} \). Now, we use convexity of \( g_0^{(j-1)} \).
above”. We first need to establish a useful inequality. Since \( \bar{g}_n^{(k-2)} \) is convex, we have for all \( t' \in [x_0 - Mn^{-1/(2k+1)}, x_0 + Mn^{-1/(2k+1)}] \) and

\[
\bar{g}_n^{(k-2)}(t') \leq \bar{g}_n^{(k-2)}(\xi_{-1}) + \frac{\bar{g}_n^{(k-2)}(\xi_1) - \bar{g}_n^{(k-2)}(\xi_{-1})}{\xi_{n,1} - \xi_{-1}}(t' - \xi_{-1}).
\]

By successive integrations of the last inequality between \( \bar{g}_n^{(j-1)}(t) \), we obtain

\[
\bar{g}_n^{(j-1)}(t) - \bar{g}_n^{(j-1)}(\xi_{-1}) \leq \bar{g}_n^{(j)}(\xi_{-1})(t - \xi_1) + \frac{\bar{g}_n^{(j)}(\xi_{-1}) + 2cn^{-2/(2k+1)}(t - \xi_{-1})^{k-j}}{2!(k-j)!}.
\]

It follows that with probability greater than \( 1 - 2\epsilon \), we have

\[
\bar{g}_n^{(j-1)}(t) \leq \bar{g}_n^{(j-1)}(\xi_{-1}) + \bar{g}_n^{(j)}(\xi_{-1})(t - \xi_1) + \frac{\bar{g}_n^{(j)}(\xi_{-1}) + 2cn^{-2/(2k+1)}(t - \xi_{-1})^{k-j}}{2!(k-j)!}.
\]

\[
\leq g_0^{(j-1)}(\xi_{-1}) + \frac{c}{K}n^{-1/(2k+1)} + \frac{(t - \xi_{-1})}{(k-j)!} + \frac{(t - \xi_{-1})}{(k-j)!}.
\]

\[
\leq \frac{\bar{g}_n^{(j-1)}(\xi_{-1}) + \bar{g}_n^{(j)}(\xi_{-1})(t - \xi_1) + \frac{\bar{g}_n^{(j)}(\xi_{-1}) + 2cn^{-2/(2k+1)}(t - \xi_{-1})^{k-j}}{2!(k-j)!}}{(t - \xi_{-1})}
\]

\[
+ \frac{\bar{g}_n^{(j-1)}(\xi_{-1}) + \bar{g}_n^{(j)}(\xi_{-1})(t - \xi_1) + \frac{\bar{g}_n^{(j)}(\xi_{-1}) + 2cn^{-2/(2k+1)}(t - \xi_{-1})^{k-j}}{2!(k-j)!}}{(t - \xi_{-1})}
\]

\[
+ \frac{\bar{g}_n^{(j-1)}(\xi_{-1}) + \bar{g}_n^{(j)}(\xi_{-1})(t - \xi_1) + \frac{\bar{g}_n^{(j)}(\xi_{-1}) + 2cn^{-2/(2k+1)}(t - \xi_{-1})^{k-j}}{2!(k-j)!}}{(t - \xi_{-1})}
\]
\[ + \left( c(1 + K^{k-j}) + c' \sum_{p=1}^{k-j} \frac{K^p}{p!} + \frac{D_1K^{k-j+1}}{k!} \right) n^{-(k-j+1)/(2k+1)} \]

\[ = g_{0}^{(j-1)}(x_0) + g_{0}^{(j)}(x_0)(t-x_0) + \cdots + g_{0}^{(k-j)}(x_0)(t-x_0)^{k-j} \frac{(k-j)!}{(k-j)!} \]

\[ + K'n^{-(k-j+1)/(2k+1)} \]

with \( K' = c(1 + K^{k-j}) + c' \sum_{p=1}^{k-j} \frac{K^p}{p!} + \frac{D_1K^{k-j+1}}{k!} \). It follows that (2.2) holds for \( j - 1 \).

\[ \blacksquare \]

**Proof of Lemma 2.2.** Fix \( K > 0 \). Recall that \( r_k \equiv 1/(2k + 1) \). We will prove the lemma for \( t \geq 0 \); similar arguments can be used for \( t \in [-K, 0) \).

We have

\[ \gamma_n^{\text{loc}}(t) = n^{2kr_k} \int_{x_0}^{x_0+tn-r_k} \cdots \int_{x_0}^{v_2} \left\{ G_n(v_1) - G_n(x_0) - \sum_{u=1}^{v_1} \left( g_0(x_0) + (u-x_0)g_0'(x_0) \right) du \right\} dv_1dv_2\ldots dv_{k-1} \]

\[ = A_n + B_n, \]

where

\[ A_n = n^{2kr_k} \int_{x_0}^{x_0+tn-r_k} \int_{x_0}^{v_2} \cdots \int_{x_0}^{v_2} \left\{ G_n(v_1) - G_n(x_0) - (G_0(v_1) - G_0(x_0)) \right\} dv_1dv_2\ldots dv_{k-1}, \]

and

\[ B_n = n^{2kr_k} \int_{x_0}^{x_0+tn-r_k} \int_{x_0}^{v_2} \cdots \int_{x_0}^{v_2} \left\{ G_0(v_1) - G_0(x_0) - \int_{x_0}^{v_1} \left( g_0(x_0) + (u-x_0)g_0'(x_0) \right) \right\} dv_1dv_2\ldots dv_{k-1}. \]
\begin{align*}
& + \cdots + \frac{1}{(k-1)!} (u - x_0)^{k-1} g_0^{(k-1)}(x_0) \int_{x_0}^{v_1} \int_{x_0}^{v_2} \int_{x_0}^{v_3} \cdots \int_{x_0}^{v_{k-1}} dv_1 dv_2 \cdots dv_{k-1}.
\end{align*}

But, with \( U_n \) denoting \( \sqrt{n}(\Gamma_n - I) \), \( \Gamma_n(t) = n^{-1} \sum_{i=1}^{n} 1_{[\xi_i \leq t]} \) where \( \xi_1, \ldots, \xi_n \) are i.i.d. \( U(0, 1) \) random variables, we have

\begin{align*}
A_n &= \frac{d}{n^{2k/r_k - 1/2}} \int_{x_0}^{x_0 + tn^{-r_k}} \int_{x_0}^{v_{k-1}} \int_{x_0}^{v_k} \cdots \int_{x_0}^{v_2} \left( U_n(G_0(v_1)) - U_n(G_0(x_0)) \right) \prod_{j=1}^{k-1} dv_j \\
&= \frac{n^{(k-1)/2} r_k}{n^{2k/r_k - 1/2}} \int_{x_0}^{x_0 + tn^{-r_k}} \int_{x_0}^{v_{k-1}} \int_{x_0}^{v_k} \cdots \int_{x_0}^{v_2} \left( U_n(G_0(v_1)) - U_n(G_0(x_0)) \right) \prod_{j=1}^{k-1} dv_j,
\end{align*}

and using Taylor expansion of \( G_0(v_1) \) in the neighborhood of \( x_0 \),

\begin{align*}
B_n &= n^{2k/r_k} \int_{x_0}^{x_0 + tn^{-r_k}} \int_{x_0}^{v_{k-1}} \int_{x_0}^{v_2} \frac{(v_1 - x_0)^{k+1}}{(k+1)!} \left( g_0^{(k)}(v_1) - g_0^{(k)}(x_0) \right) \\
&\quad \cdot \prod_{i=1}^{k-1} dv_i \\
&+ n^{2k/r_k} \int_{x_0}^{x_0 + tn^{-r_k}} \int_{x_0}^{v_{k-1}} \int_{x_0}^{v_2} \frac{(v_1 - x_0)^{k+1}}{(k+1)!} g_0^{(k)}(x_0) \prod_{i=1}^{k-1} dv_i \\
&= B_{n1} + B_{n2},
\end{align*}

where \(|v^*_1 - x_0| \leq |v_1 - x_0|\). Now,

\begin{align*}
\frac{B_{n2}}{g_0^{(k)}(x_0)} &= \frac{n^{2k/r_k}}{(k+2)!} \int_{x_0}^{x_0 + tn^{-r_k}} \int_{x_0}^{v_{k-1}} \int_{x_0}^{v_2} \frac{(v_2 - x_0)^{k+2}}{(k+2)!} dv_2 \cdots dv_{k-1} \\
&= \frac{n^{2k/r_k}}{(k+3)!} \int_{x_0}^{x_0 + tn^{-r_k}} \int_{x_0}^{v_{k-1}} \int_{x_0}^{v_3} \frac{(v_3 - x_0)^{k+3}}{(k+3)!} dv_3 \cdots dv_{k-1} \\
&\vdots
\end{align*}
Furthermore, by continuity of $g^{(k)}_0$ at $x_0$, we deduce that $B_n(t) = o(1)$ uniformly in $0 \leq t \leq K$ and hence

\[(4.36) \quad B_n \to \frac{1}{(2k)!} g^{(k)}_0(x_0)t^{2k},\]

as $n \to \infty$ uniformly in $0 \leq t \leq K$. Using the identity

$$\mathbb{U}(G_0(v)) - \mathbb{U}(G_0(x_0)) = W(G_0(v)) - W(G_0(x_0)) - (G_0(v) - G_0(x_0))W(1),$$

where $W$ is two-sided Brownian motion process, we have

$$A_n = n^{(k-1/2)r_k} \int_{x_0}^{x_0+tn-r_k} \int_{x_0}^{v_{k-1}} \cdots \int_{x_0}^{v_2} \left( \mathbb{U}_n(v_1) - \mathbb{U}(v_1) - (\mathbb{U}_n(x_0) - \mathbb{U}(x_0)) \right) dv_1 \ldots dv_{k-1}
+ n^{(k-1/2)r_k} \int_{x_0}^{x_0+tn-r_k} \int_{x_0}^{v_{k-1}} \cdots \int_{x_0}^{v_2} \left( W(G_0(v)) - W(G_0(x_0)) \right) dv_1 \ldots dv_{k-1}
- W(1)n^{(k-1/2)r_k} \int_{x_0}^{x_0+tn-r_k} \int_{x_0}^{v_{k-1}} \cdots \int_{x_0}^{v_2} (G_0(v_1) - G_0(x_0)) dv_1 \ldots dv_{k-1}
= A_{n1} + A_{n2} + A_{n3}.$$

But,

$$A_{n1} \leq 2n^{(k-1/2)r_k} \|\mathbb{U}_n - \mathbb{U}\|_\infty \int_{x_0}^{x_0+tn-r_k} \int_{x_0}^{v_{k-1}} \cdots \int_{x_0}^{v_2} dv_1 \ldots dv_{k-1}
= 2n^{(k-1/2)r_k} \|\mathbb{U}_n - \mathbb{U}\|_\infty \int_{x_0}^{x_0+tn-r_k} \int_{x_0}^{v_{k-1}} \cdots \int_{x_0}^{v_3} (v_2 - x_0) dv_2 \ldots dv_{k-1}
= 2n^{(k-1/2)r_k} \|\mathbb{U}_n - \mathbb{U}\|_\infty \int_{x_0}^{x_0+tn-r_k} \int_{x_0}^{v_{k-1}} \cdots \int_{x_0}^{v_1} \frac{1}{2} (v_3 - x_0)^2 dv_3
\cdots$$
\[ A_{n3} \leq |W(1)|g_0(x_0)n^{(k-1)/2}r_k \int_{x_0}^{x_0+tn^{-r_k}} \int_{x_0}^{v_k-1} \cdots \int_{x_0}^{v_2} (v_1 - x_0)dv_1 \cdots dv_{k-1} \]
\[ = |W(1)|g_0(x_0)n^{(k-1)/2}r_k \int_{x_0}^{x_0+tn^{-r_k}} \int_{x_0}^{v_k-1} \cdots \int_{x_0}^{v_2} \frac{1}{2}(v_1 - x_0)^2dv_2 \]
\[ \vdots \]
\[ = |W(1)|g_0(x_0)n^{(k-1)/2}r_k \frac{1}{(k-1)!} \int_{x_0}^{x_0+tn^{-r_k}} (v_{k-1} - x_0)^{k-1}dv_{k-1} \]
\[ = |W(1)|g_0(x_0)n^{(k-1)/2}r_k \frac{1}{k!} \left( \frac{t}{n^{rk}} \right)^k \]
\[ \text{(4.38) } |W(1)|g_0(x_0)t^{k-n^{-rk}/2} \rightarrow_p 0, \]

as \( n \rightarrow \infty \) uniformly in \( 0 \leq t \leq K \). Finally, using the change of variables \( s_j = n^{1/(2k+1)}(v_j - x_0) = n^{rk}(v_j - x_0) \) for \( j = 1, \ldots, k-1 \), we have
\[ A_{n2} = n^{(k-1)/2}r_k \int_{x_0}^{x_0+tn^{-r_k}} \int_{x_0}^{v_k-1} \cdots \int_{x_0}^{v_2} \left( W(G_0(v_1)) - W(G_0(x_0)) \right) \]
\[ dv_1 \cdots dv_{k-1} \]
\[ = n^{(k-1)/2}r_k n^{-(k-1)r_k} \int_{0}^{t} \int_{0}^{s_{k-1}} \cdots \int_{0}^{s_2} \left( W(G_0(n^{-rk}s_1 + x_0)) - W(G_0(x_0)) \right) \]
\[ ds_1 \cdots ds_{k-1} \]
\[ \frac{d}{ds} n^{r_k/2} \int_0^t \int_0^{s_{k-1}} \cdots \int_0^{s_2} W \left( G_0(n^{-r_k} s_{1} + x_0) - G_0(x_0) \right) ds_1 \cdots ds_k \]

\[ = \int_0^t \int_0^{s_{k-1}} \cdots \int_0^{s_2} W \left( n^{r_k} (G_0(n^{-r_k} s_{1} + x_0) - G_0(x_0)) \right) ds_1 \cdots ds_k \]

\[ \rightarrow \int_0^t \int_0^{s_{k-1}} \cdots \int_0^{s_2} W(s_1 g_0(x_0)) ds_1 \cdots ds_k \quad \text{as } n \to \infty \]

\[ \text{(4.39)} \]

\[ \sqrt{g_0(x_0)} \int_0^t \int_0^{s_{k-1}} \cdots \int_0^{s_2} W(s_1) ds_1 \cdots ds_k. \]

Therefore, combining (4.36), (4.37), (4.38) and (4.39) yields

\[ \mathbb{W}^n_{\text{loc}}(t) \Rightarrow \sqrt{g_0(x_0)} \int_0^t \int_0^{s_{k-1}} \cdots \int_0^{s_2} W(s_1) ds_1 \cdots ds_k + \frac{1}{(2k)!} t^{2k} g_0^{(k)}(x_0) \]

\[ \equiv Y_{a,\sigma}(t) \]

for \( 0 \leq t \leq K \). A similar argument for \(-K \leq t < 0\) yields the conclusion.\[ \blacksquare \]

**Proof of Lemma 2.3.** We apply the same arguments in the proof of Lemma 2.2 in the case of the LSE. \[ \blacksquare \]

**Proof of Lemma 2.5.** Groeneboom, Jongbloed and Wellner (Groeneboom, Jongbloed, and Wellner (2001b)) chose the “canonical process” to be

\[ Y(t) = \int_0^t W(y) dy + t^4, \]

so that with \( X(t) = Y'(t) = W(t) + 4t^3 \) we have

\[ dX(t) = 12t^2 dt + dW(t) \equiv f_0(t) dt + dW(t) \]

where \( f_0(t) = 12t^2 \) is convex. Here we make a different choice, namely \( f_0(t) = (-1)^k t^k \) (so that \( f_0(t) = t^2 \) in the case \( k = 2 \)). Thus we will rescale the limiting process \( Y_{a,\sigma} \) so that we obtain the “canonical process”

\[ (4.4Y)(t) = \int_0^t \int_0^{u_{k-1}} \cdots \int_0^{u_2} W(u_1) du_1 du_2 \cdots du_{k-1} + (-1)^k \frac{k!}{(2k)!} t^{2k}, \]

for \( t \geq 0 \). Let \( \sigma = \sqrt{g_0(x_0)} \) and \( a = (-1)^k g_0^{(k)}(x_0)/k! \). Then

\[ Y_{a,\sigma}(t) = \sqrt{g_0(x_0)} \int_0^t \int_0^{u_{k-1}} \cdots \int_0^{u_2} W(u_1) du_1 \cdots du_{k-1} \]

\[ + (-1)^k \frac{k!}{(2k)!} t^{2k}. \]
Furthermore,

\[
Y_{a,\sigma}(t) = \left\{ \begin{array}{l}
a(-1)^k \frac{k!}{(2k)!} t^{2k} + \sigma \int_0^t \int_0^{u_{k-1}} \cdots \int_0^{u_2} W(u_1) du_1 \cdots du_{k-1} \\
= a(-1)^k \frac{k!}{(2k)!} t^{2k} + \alpha^{1/2} \sigma \int_0^t \int_0^{u_{k-1}} \cdots \int_0^{u_2} W(\alpha u_1) du_1 \cdots du_{k-1} \\
= a(-1)^k \frac{k!}{(2k)!} t^{2k} \\
+ \alpha^{-1/2} \sigma \int_0^t \int_0^{u_{k-1}} \cdots \int_0^{\alpha u_2} \frac{1}{\alpha} W(u_1) du_1 \cdots du_{k-1} \\
= a(-1)^k \frac{k!}{(2k)!} t^{2k} \\
+ \alpha^{-1/2} \sigma \int_0^t \int_0^{u_{k-1}} \cdots \int_0^{\alpha u_3} \frac{1}{\alpha^2} W(u_1) du_1 \cdots du_{k-1} \\
= \vdots \\
= a(-1)^k \frac{k!}{(2k)!} t^{2k} + \frac{\sigma}{\sqrt{\alpha}} \int_0^t \int_0^{u_{k-1}} \cdots \int_0^{u_2} W(u_1) \frac{1}{\alpha^{k-1}} du_1 \cdots du_{k-1} \\
= a(-1)^k \frac{k!}{(2k)!} t^{2k} + \alpha^{1/2-k} \sigma \int_0^t \int_0^{u_{k-1}} \cdots \int_0^{u_2} W(u_1) du_1 \cdots du_{k-1}.
\end{array} \right.
\]

Therefore,

\[
s_1 Y_{a,\sigma}(s_2 t) \overset{d}{=} a(-1)^k \frac{k!}{(2k)!} s_1(s_2 t)^{2k} \\
+ s_1 \alpha^{1/2-k} \sigma \int_0^t \int_0^{s_2 \alpha t} \cdots \int_0^{u_2} W(u_1) du_1 \cdots du_{k-1},
\]

and the process on the right side of the last display equals \(Y_k\) as defined in (4.41) if \(as_1s_2^{2k} = 1\), \(s_1 \alpha^{1/2-k} \sigma = 1\), and \(s_2 \alpha = 1\). Solving this system of equations yields \(\alpha = (a/\sigma)^{2/(2k+1)}\), and therefore \(s_1\) and \(s_2\) are given by (2.3) and (2.4) respectively. Thus, \(Y_{a,\sigma}(t) \overset{d}{=} Y_k(t/s_2)/s_1\) \hfill \Box

**Proof of Lemma 2.4.** We now consider the difference of the two local processes \(\hat{\gamma}_n^{\text{loc}}\) and \(\hat{H}_n^{\text{loc}}\). We have

\[
\hat{H}^{\text{loc}}_n(t) - \hat{\gamma}^{\text{loc}}_n(t) \\
= n^{2kr} \int_{x_0}^{x_0+tn^{-r_k}} \int_{x_0}^{v_{k-1}} \cdots \int_{x_0}^{v_2} \left\{ \left( \hat{G}_n(v_1) - \hat{G}_n(x_0) \right) \right\}
\]
\[ - (G_n(v_1) - G_n(x_0)) \]
\[ dv_1 \cdots dv_{k-1} \] 
\[ + \tilde{A}_{(k-1)n} t^{k-1} + \cdots + \tilde{A}_{1n} t + \tilde{A}_{0n} \]
\[ = n^{2kr_k} \int_{x_0}^{x_0 + tn^{-r_k}} \int_{x_0}^{v_k-1} \int_{x_0}^{v_2} (\tilde{G}_n(v_1) - G_n(v_1)) \, dv_1 \cdots dv_{k-1} \]
\[ - \frac{n^{(k+1)r_k}}{(k-1)!} \left( \tilde{G}_n(x_0) - G_n(x_0) \right) t^{k-1} \]
\[ + \tilde{A}_{(k-1)n} t^{k-1} + \cdots + \tilde{A}_{1n} t + \tilde{A}_{0n} \]
\[ = n^{2kr_k} \int_{x_0}^{x_0 + tn^{-r_k}} \int_{x_0}^{v_k-1} \int_{x_0}^{v_2} \left( \tilde{G}_n(v_1) - G_n(v_1) \right) \, dv_1 \cdots dv_{k-1} \]
\[ - \tilde{A}_{(k-1)n} t^{k-1} + \tilde{A}_{(k-1)n} t^{k-1} + \cdots + \tilde{A}_{1n} t + \tilde{A}_{0n} \]
\[ = n^{2kr_k} \int_{x_0}^{x_0 + tn^{-r_k}} \int_{x_0}^{v_k-1} \int_{x_0}^{v_2} \left( \tilde{G}_n(v_1) - G_n(v_1) \right) \, dv_1 \cdots dv_{k-1} \]
\[ - n^{2kr_k} \int_{x_0}^{x_0 + tn^{-r_k}} \int_{x_0}^{v_k-1} \int_{x_0}^{v_3} \, dv_2 \cdots dv_{k-1} \]
\[ \cdot \int_{x_0}^{v_2} \left( \tilde{G}_n(v_1) - G_n(v_1) \right) \, dv_1 \]
\[ + \tilde{A}_{(k-2)n} t^{k-2} + \cdots + \tilde{A}_{1n} t + \tilde{A}_{0n} \]
\[ = n^{2kr_k} \int_{x_0}^{x_0 + tn^{-r_k}} \int_{x_0}^{v_k-1} \int_{x_0}^{v_2} \left( \tilde{G}_n(v_1) - G_n(v_1) \right) \, dv_1 \cdots dv_{k-1} \]
\[ - n^{(k+2)r_k} \frac{t^{k-2}}{(k-2)!} \int_{x_0}^{x_0} \left( \tilde{G}_n(v_1) - G_n(v_1) \right) \, dv_1 + \tilde{A}_{(k-2)n} t^{k-2} \]
\[ + \tilde{A}_{(k-3)n} t^{k-3} + \cdots + \tilde{A}_{1n} t + \tilde{A}_{0n} \]
\[ = n^{2kr_k} \int_{x_0}^{x_0 + tn^{-r_k}} \int_{x_0}^{v_k-1} \int_{x_0}^{v_2} \left( \tilde{G}_n(v_1) - G_n(v_1) \right) \, dv_1 \cdots dv_{k-1} \]
\[ - \tilde{A}_{(k-2)n} t^{k-2} + \tilde{A}_{(k-2)n} t^{k-2} + \cdots + \tilde{A}_{1n} t + \tilde{A}_{0n} \]
\[ = n^{2kr_k} \int_{x_0}^{x_0 + tn^{-r_k}} \int_{x_0}^{v_k-1} \int_{x_0}^{v_2} \left( \tilde{G}_n(v_1) - G_n(v_1) \right) \, dv_1 \cdots dv_{k-1} \]
\[ + \tilde{A}_{(k-3)n} t^{k-2} + \cdots + \tilde{A}_{1n} t + \tilde{A}_{0n} \]


by the first Fenchel condition satisfied by the LSE.

Similarly, for the localized processes \( \widehat{\nu}_n^{\text{loc}} \) and \( \widehat{H}_n^{\text{loc}} \), by the particular choice of \( \widehat{A}_{jn}, 0 \leq j \leq k - 1 \), we have

\[
(\widehat{H}_n^{\text{loc}}(t) - \widehat{\nu}_n^{\text{loc}}(t))/g_0(x_0)
= n^{2kr} \int_{x_0}^{x_0 + tn^{-r_k}} \cdots \int_{x_0}^{x_0 + tn^{-r_k}} \frac{\hat{g}_n(v) - g_0(v)}{g_n(v)} dv_1 \cdots dv_{k-1}
- n^{2kr} \int_{x_0}^{v_{k-1}} \cdots \int_{x_0}^{v_1} \frac{1}{g_n(v)} d(G_n - G_0)(v) dv_1 \cdots dv_{k-1}
+ \widehat{A}_{(k-1)n} k^{k-1} + \cdots \widehat{A}_{0n}
= n^{2kr} (\frac{k}{k!}) n^{-kr} \int_{x_0}^{x_0 + tn^{-r_k}} \cdots \int_{x_0}^{x_0 + tn^{-r_k}} \frac{1}{g_n(v)} dG_n(v) \Pi_{i=1}^{k-1} dv_i
+ \widehat{A}_{(k-1)n} k^{k-1} + \cdots \widehat{A}_{0n}.
\]

But notice that for any \( t \geq 0 \)

\[
\int_0^t \frac{1}{g_n(u)} dG_n(u) = \frac{1}{(k-1)!} \widehat{H}_n^{(k-1)}(t).
\]

It follows that

\[
\int_{x_0}^{x_0 + tn^{-r_k}} \cdots \int_{x_0}^{x_0 + tn^{-r_k}} \frac{1}{g_n(v)} dG_n(v) dv_1 \cdots dv_{k-1}
= \frac{1}{(k-1)!} \int_{x_0}^{x_0 + tn^{-r_k}} \cdots \int_{x_0}^{x_0 + tn^{-r_k}} (\widehat{H}_n^{(k-1)}(v_1) - \widehat{H}_n^{(k-1)}(x_0)) dv_1 \cdots dv_{k-1}
= \frac{1}{(k-1)!} \left( \widehat{H}_n(x_0 + tn^{-r_k}) - \sum_{j=0}^{k-1} \frac{t^{j} n^{-j r_k}}{j!} \widehat{H}_n^{(j)}(x_0) \right).
\]

Therefore,

\[
\widehat{H}_n^{\text{loc}}(t) - \widehat{\nu}_n^{\text{loc}}(t)
= n^{2k/(2k+1)} g_0(x_0) \left\{ - \frac{\widehat{H}_n(x_0 + tn^{-1/(2k+1)})}{(k-1)!} + \frac{t^k}{k!} n^{-k/(2k+1)} \right\}
\]
successive jump points,
\( \tau = j \) \((2.5)\) for \( j = 1 \) and \( k - 2 \) and then use induction for \( 2 \leq j \leq k - 3 \). Proving
\((2.5)\) for \( j = k - 1 \) would have been sufficient but we wanted to show it for \( j = k - 2 \) to give a better idea about how the proof works. Now consider \( k \) successive jump points, \( \tau_1, \ldots, \tau_k \), of \( g_{n(k-1)} \) where \( \tau_1 \) is the first jump after \( x_0 \). By the mean value theorem, there exist \( \tau_i^{(1)} \in (\tau_1, \tau_2), \tau_2^{(1)} \in (\tau_2, \tau_3), \ldots, \tau_{k-1}^{(1)} \in (\tau_{k-1}, \tau_k) \) such that \( \Delta_n' (\tau_i^{(1)}) = 0 \) for \( 1 \leq i \leq k - 1 \). Also, by the same theorem there exist \( \tau_i^{(2)} \in (\tau_1^{(1)}, \tau_2^{(1)}), \ldots, \tau_{k-2}^{(2)} \in (\tau_{k-2}^{(1)}, \tau_{k-1}^{(1)}) \) such that \( \Delta_n'' (\tau_i^{(2)}) = 0 \) for \( 1 \leq i \leq k - 2 \). It is easy to see that we can carry on

Proof of Lemma 2.6. We will give the proof only for the LSE as the arguments are very similar for the MLE. Let \( j \in \{0, \ldots, k - 1\} \) and denote \( \hat{\Delta}_n(x) = \hat{H}_n(x) - \hat{Y}_n(x) \) for all \( x \geq 0 \). We will start by proving (2.5) for \( j = k - 1 \) and \( k - 2 \) and then use induction for \( 2 \leq j \leq k - 3 \). Proving
\((2.5)\) for \( j = k - 1 \) would have been sufficient but we wanted to show it for \( j = k - 2 \) to give a better idea about how the proof works. Now consider \( k \) successive jump points, \( \tau_1, \ldots, \tau_k \), of \( g_{n(k-1)} \) where \( \tau_1 \) is the first jump after \( x_0 \). By the mean value theorem, there exist \( \tau_i^{(1)} \in (\tau_1, \tau_2), \tau_2^{(1)} \in (\tau_2, \tau_3), \ldots, \tau_{k-1}^{(1)} \in (\tau_{k-1}, \tau_k) \) such that \( \Delta_n' (\tau_i^{(1)}) = 0 \) for \( 1 \leq i \leq k - 1 \). Also, by the same theorem there exist \( \tau_i^{(2)} \in (\tau_1^{(1)}, \tau_2^{(1)}), \ldots, \tau_{k-2}^{(2)} \in (\tau_{k-2}^{(1)}, \tau_{k-1}^{(1)}) \) such that \( \Delta_n'' (\tau_i^{(2)}) = 0 \) for \( 1 \leq i \leq k - 2 \). It is easy to see that we can carry on
this reasoning up to the \((k-1)\)-st level of differentiation and so there exists \(\tau^{(k-1)}\) such that
\[
\tilde{\Delta}_n^{(k-1)}(\tau^{(k-1)}) = 0.
\]
Denote \(\tau = \tau^{(k-1)}\). We can write
\[
\tilde{\Delta}_n^{(k-1)}(x_0) = \tilde{\Delta}_n^{(k-1)}(x_0) - \tilde{\Delta}_n^{(k-1)}(\tau).
\]
But since
\[
\tilde{\Delta}_n^{(k-1)}(x) = \int_0^x d(\tilde{G}_n(t) - G_n(t)), \text{ for } x \geq 0,
\]
we can write,
\[
|\tilde{\Delta}_n^{(k-1)}(x_0)| = \left| \int_{x_0}^{x_0} d(\tilde{G}_n(t) - G_n(t)) \right|
\leq \left| \int_{x_0}^{\tau} d(\tilde{G}_n(t) - G_0(t)) \right| + \left| \int_{x_0}^{\tau} d(G_n(t) - G_0(t)) \right|
= \left| \int_{x_0}^{\tau} (\tilde{g}_n(t) - g_0(t)) dt \right| + \left| \int_{x_0}^{\tau} d(G_n(t) - G_0(t)) \right|
\leq \int_{x_0}^{\tau} |\tilde{g}_n(t) - g_0(t)| dt + \left| \int_{x_0}^{\tau} d(G_n(t) - G_0(t)) \right|.
\]
Fix \(0 < \epsilon < 1\). By Lemma 2.1 and Proposition 2.1, we can find \(M > 0\) and \(c > 0\) such that with probability greater than \(1 - \epsilon\)
\[
x_0 \leq \tau \leq x_0 + Mn^{-1/(2k+1)}
\]
and
\[
|\tilde{g}_n(t) - g_0(x_0) - g_0'(x_0)(t-x_0) - \cdots - \frac{g_0^{(k-1)}(x_0)}{(k-1)!}(t-x_0)^{k-1}| \leq cn^{-k/(2k+1)}
\]
for \(x_0 - Mn^{-1/(2k+1)} \leq t \leq x_0 + Mn^{-1/(2k+1)}\). On the other hand, using Taylor expansion, we can find \(d > 0\) that
\[
|g_0(t) - g_0(x) + g_0'(x_0)(t-x_0) - \cdots - \frac{g_0^{(k-1)}(x_0)}{(k-1)!}(t-x_0)^{k-1}| \leq d (t-x_0)^k
\]
\[
\leq c'n^{-k/(2k+1)}
\]
for \( x_0 - Mn^{-1/(2k+1)} \leq t \leq x_0 + Mn^{-1/(2k+1)} \) and where \( c' = dM^k \). It follows that
\[
\int_{x_0}^{\tau} |\tilde{g}_n(t) - g_0(t)| \, dt \leq (c + c')n^{-k/(2k+1)} \int_{x_0}^{\tau} dt = (c + c')n^{-k/(2k+1)} \times (\tau - x_0) \leq (c + c')Mn^{-(k+1)/(2k+1)}.
\]

To finish off the proof, we only need to check that
\[
\left| \int_{x_0}^{\tau} d(G_n(t) - G_0(t)) \right| = O_p(n^{-(k+1)/(2k+1)}).
\]

But this can be shown using similar arguments to those in the proof of Proposition 4.1. Indeed,
\[
\int_{x_0}^{\tau} d(G_n(t) - G_0(t)) = \int_{\tau}^{\infty} 1_{[x_0, \tau]}(t) d(G_n(t) - G_0(t)),
\]
so consider now the empirical process
\[
U_n(y, z) = \int_{0}^{\infty} 1_{[y, z]}(t) d(G_n(t) - G_0(t))
\]
for \( 0 < y \leq z \) and the class of functions
\[
\mathcal{F}_{y, R} = \{ f_{y, z} : f_{y, z}(t) = 1_{[y, z]}(t), y \leq z \leq y + R \}
\]
for a fixed \( y > 0 \) and \( R > 0 \). By application of lemma 5.1 with \( d = 1 \) and \( \alpha = k \), it follows that for each \( \epsilon > 0 \) there exist \( \delta > 0 \) and \( R > 0 \) such that
\[
|U_n(y, z)| \leq \epsilon(z - y)^{k+1} + O_p(n^{-(k+1)/(2k+1)})
\]
for all \( |y - x_0| \leq \delta, z \in [y, y + R] \). Thus we conclude that
\[
\left| \int_{x_0}^{\tau} d(G_n(t) - G_0(t)) \right| = o_p \left((\tau - x_0)^{k+1}\right) + O_p(n^{-(k+1)/(2k+1)}) = O_p((n^{-(k+1)/(2k+1)})
\]
and the result follows for \( j = k - 1 \). Note that we obtain the same result if we replace \( x_0 \) by any \( x \) in an neighborhood of \( x_0 \) of the form \( |x_0 -
\( M n^{-1/(2k+1)}, x_0 + M n^{-1/(2k+1)} \), for some constant \( M > 0 \); i.e., we can find \( K > 0 \) independent of \( x \) such that
\[
\left| \tilde{\Delta}_n^{(k-1)}(x) \right| \leq K n^{-(k+1)/(2k+1)}
\]
with large probability. Now, let \( j = k - 2 \). We have,
\[
\tilde{\Delta}_n^{(k-2)}(x_0) = \int_0^{x_0} (x_0 - t) d(\tilde{G}_n(t) - \mathbb{G}_n(t)).
\]
Let \( \tau \) be a zero of \( \tilde{\Delta}_n^{(k-2)} \) (we can find such a zero the same way we did for \( \tilde{\Delta}_n^{(k-1)} \)). We can write
\[
\tilde{\Delta}_n^{(k-2)}(x_0) = \tilde{\Delta}_n^{(k-2)}(x_0) - \tilde{\Delta}_n^{(k-2)}(\tau)
\]
\[
= \int_0^{x_0} (x_0 - t) d(\tilde{G}_n(t) - \mathbb{G}_n(t)) - \int_0^{\tau} (\tau - t) d(\tilde{G}_n(t) - \mathbb{G}_n(t))
\]
\[
= - \int_{x_0}^{\tau} (x_0 - t) d(\tilde{G}_n(t) - \mathbb{G}_n(t)) - (\tau - x_0) \int_0^{\tau} d(\tilde{G}_n(t) - \mathbb{G}_n(t))
\]
\[
= - \int_{x_0}^{\tau} (x_0 - t) d(\tilde{G}_n(t) - \mathbb{G}_n(t)) - (\tau - x_0) \tilde{\Delta}_n^{(k-1)}(\tau).
\]
Let \( M > 0 \) be such that \( x_0 \leq \tau \leq x_0 + M n^{-1/(2k+1)} \). By the previous result, there exists \( c > 0 \) such that
\[
\left| (\tau - x_0) \tilde{\Delta}_n^{(k-1)}(\tau) \right| \leq c n^{-(k+2)/(2k+1)}
\]
with large probability. Now,
\[
\left| \int_{x_0}^{\tau} (x_0 - t) d(\tilde{G}_n(t) - \mathbb{G}_n(t)) \right| \leq \left| \int_{x_0}^{\tau} (t - x_0)|\tilde{g}_n(t) - g_0(t)|dt \right|
\]
\[
+ \left| \int_{x_0}^{\tau} (t - x_0)d(\mathbb{G}_n(t) - G_0(t)) \right|
\]
We can find \( d > 0 \) such that
\[
|\tilde{g}_n(t) - g_0(x_0) - g_0'(x_0)(t - x_0) - \cdots - \frac{g_0^{(k-1)}(x_0)}{(k-1)!}(t - x_0)^{k-1}| \leq dn^{-k/(2k+1)}
\]
and
\[
|g_0(t) - g_0(x_0) - g_0'(x_0)(t - x_0) - \cdots - \frac{g_0^{(k-1)}(x_0)}{(k-1)!}(t - x_0)^{k-1}| \leq dn^{-k/(2k+1)}
\]
for all $t \in [x_0 - M n^{-1/(2k+1)}, x_0 + M n^{-1/(2k+1)}]$ with large probability. It follows that
\[
\int_{x_0}^{\tau} (t - x_0)|\tilde{g}_n(t) - g_0(t)|dt \leq 2d n^{-k/(2k+1)} \int_{x_0}^{\tau} (t - x_0)dt
\]
\[
= d n^{-k/(2k+1)} (\tau - x_0)^2
\]
\[
\leq 4dM^2 n^{-(k+2)/(2k+1)}.
\]
with large probability. Finally, via empirical processes arguments and lemma 5.1 with $d = 2$, it follows that
\[
\left| \int_{x_0}^{\tau} (t - x_0)(G_n(t) - G_0(t)) \right| = O_p(n^{-(k+2)/(2k+1)})
\]
and the result follows for $j = k - 2$. The same result holds if we replace $x_0$ by any $x \in [x_0 - M n^{-1/(2k+1)}, n^{-1/(2k+1)}, x_0 + M n^{-1/(2k+1)}]$, for some $M > 0$; i.e., we can find $K > 0$ independent of $x$ such that
\[
\left| \tilde{\Delta}_n^{(k-2)}(x) \right| \leq Kn^{-(k+2)/(2k+1)}
\]
with large probability. Now let $0 \leq j \leq k - 3$ and fix $\epsilon > 0$. Suppose that for all $j' > j$ and $M > 0$, there exists $c > 0$ such that for all $z \in [x_0 - M n^{-1/(2k+1)}, x_0 + M n^{-1/(2k+1)}]$,
\[
(k - 1 - j')! |\tilde{\Delta}_n^{(j')}(z)| \leq cn^{-(2k-j')/(2k+1)}
\]
with probability greater than $1 - \epsilon$. We can write
\[
(k - 1 - j)! \tilde{\Delta}_n^{(j)}(y)
\]
\[
= \int_{0}^{y} (y - t)^{k-1-j} d(\tilde{G}_n(t) - \mathbb{G}_n(t))
\]
\[
= \int_{0}^{y} ((y - x) + (x - t))^{k-1-j} d(\tilde{G}_n(t) - \mathbb{G}_n(t))
\]
\[
= \sum_{l=0}^{k-1-j} \binom{k-1-j}{l} (y - x)^l \int_{0}^{y} (x - t)^{k-1-j-l} d(\tilde{G}_n(t) - \mathbb{G}_n(t))
\]
\[
= \sum_{l=1}^{k-1-j} \binom{k-1-j}{l} (y - x)^l \int_{0}^{y} (x - t)^{k-1-j-l} d(\tilde{G}_n(t) - \mathbb{G}_n(t))
\]
\[ + \int_0^y (x-t)^{k-1-j} d(\hat{G}_n(t) - \mathbb{G}_n(t)) \]

\[ = \sum_{l=1}^{k-1-j} \binom{k-1-j}{l} (y-x)^l \hat{\Delta}_n^{(j+l)}(y) \]

\[ + \hat{\Delta}_n^{(j)}(x) + \int_x^y (x-t)^{k-1-j} d(\hat{G}_n(t) - \mathbb{G}_n(t)). \]

Take \( x \) to be a zero of \( \hat{\Delta}_n^{(j)} \) (such a zero can be constructed using the mean value theorem as we did for \( j = k - 2 \) and \( j = k - 1 \)). Thus there exists \( M > 0 \) such that \( x_0 - Mn^{-1/(2k+1)} \leq x \leq x_0 + Mn^{-1/(2k+1)} \). Now by applying the induction hypothesis, there exists \( c > 0 \) such that for all \( y \in [x_0 - Mn^{-1/(2k+1)}, x_0 + Mn^{-1/(2k+1)}] \), we have

\[ \left| (k-1-j)! \hat{\Delta}_n^{(j)}(y) \right| \leq c \sum_{l=1}^{k-1-j} \binom{k-1-j}{l} |y-x|^l n^{-(2k-(j+l))/(2k+1)} \]

\[ + \left| \int_x^y (x-t)^{k-1-j} d(\hat{G}_n(t) - \mathbb{G}_n(t)) \right|. \]

But,

\[ \sum_{l=1}^{k-1-j} \binom{k-1-j}{l} |y-x|^l n^{-(2k-(j+l))/(2k+1)} \]

\[ \leq \left( \sum_{l=1}^{k-1-j} \binom{k-1-j}{l} (2M)^l \right) n^{-(2k-j)/(2k+1)} \]

and

\[ \left| \int_x^y (x-t)^{k-1-j} d(\hat{G}_n(t) - \mathbb{G}_n(t)) \right| = O_p(n^{-(2k-j)/(2k+1)}) \]

by using empirical processes arguments. Therefore, the result holds for \( j \) and hence for all \( j = 0, \ldots, k - 1 \).

\[ \blacksquare \]

**Proof of Lemma 2.7.** The arguments are very similar to those used in Groeneboom, Jongbloed and Wellner (Groeneboom, Jongbloed, and
We prove the lemma for \( \tilde{H}_n^l \) as the arguments are similar for \( \tilde{H}_n^l \). Let \( c > 0 \). On \([-c, c]\), define the vector-valued stochastic process

\[
Z_n(t) = \left( \tilde{H}_n^l(t), \ldots, (\tilde{H}_n^l)^{(2k-2)}(t), \mathcal{Y}_n^l(t), \ldots, (\mathcal{Y}_n^l)^{(k-2)}(t), (\tilde{H}_n^l)^{(2k-1)}(t), (\mathcal{Y}_n^l)^{(k-1)}(t) \right).
\]

This stochastic process belongs to the space

\[
E_k[-c, c] = (C[-c, c])^{3k-2} \times (D[-c, c])^2
\]

where \( C[-c, c] \) and \( D[-c, c] \) are respectively the space of continuous and right-continuous functions on \([-c, c]\). We endow the space \( E_k[-c, c] \) with the product topology induced by the uniform topology on \( C[-c, c] \) and the Skorohod topology on \( D[-c, c] \). By Proposition 2.1 and Lemma 2.6, we know that \((\tilde{H}_n^l)^{(j)}\) is tight in \( C[-c, c] \) for \( j = 0, \ldots, 2k-2 \). It follows from the same lemma together with the monotonicity of \((\tilde{H}_n^l)^{(2k-1)}\) that the latter is tight in \( D[-c, c] \). On the other hand, since the processes \((\mathcal{Y}_n^l, \ldots, (\mathcal{Y}_n^l)^{(k-2)})\) and \((\mathcal{Y}_n^l)^{(k-1)}\) converge weakly, they are tight in \((C[-c, c])^{k-1}\) and \( D[-c, c] \) respectively. Now, for a fixed \( \epsilon > 0 \), there exists an \( M > 0 \) such that with probability greater than \( 1 - \epsilon \), the process \( Z_n \) belongs to \( E_{k,M}[-c, c] \) where \( E_{k,M} = (C_M[-c, c])^{3k-2} \times (D_M[-c, c])^2 \), and \( C_M[-c, c] \) and \( D_M[-c, c] \) are respectively the subset of functions in \( C[-c, c] \) and the subset of monotone functions in \( D[-c, c] \) that are bounded by \( M \). Since the subspace \( E_{k,M}[-c, c] \) is compact, we can extract from any arbitrary sequence \( \{Z_n\} \) a further subsequence \( \{Z_{n'}\} \) that is weakly converging to some process

\[
Z_0 = \left( H_0, \ldots, H_0^{(2k-2)}, Y_0, \ldots, Y_0^{(k-2)}, H_0^{(2k-1)}, Y_0^{(k-1)} \right)
\]

in \( E_k[-c, c] \) and where \( Y_0 = Y_k \). Now, consider the functions \( \phi_1 \) and \( \phi_2 : E_k[-c, c] \to \mathbb{R} \) defined by

\[
\phi_1(z_1, \ldots, z_{3k}) = \inf_{t \in [-c, c]} (z_1(t) - z_{2k}(t)) \land 0
\]

and

\[
\phi_2(z_1, \ldots, z_{3k}) = \int_{-c}^c (z_1(t) - z_{2k}(t))dz_{3k-1}(t).
\]
It is easy to check that the functions $\phi_1$ and $\phi_2$ are both continuous. By the continuous mapping theorem, it follows that $\phi_1(Z_0) = \phi_2(Z_0) = 0$ since $\phi_1(Z_{n'}) = \phi_2(Z_{n'}) = 0$ and therefore,

$$H_0(t) \geq Y_k(t),$$

for all $t \in [-c, c]$ and

$$\int_{-c}^{c} (H_0(t) - Y_k(t))dH_0^{(2k-1)}(t) = 0.$$

It is easy to see check that $(-1)^k H_0^{(2k-2)}$ is convex. Since $c > 0$ is arbitrary, we see that $H_0$ satisfies conditions (i) and (iii) of Theorem 2.1. Furthermore, outside the interval $[-c, c]$ we can take $\tilde{H}_k$ and $\hat{Y}_k$ to be identically 0. With this choice, the condition (iv) of Theorem 2.1 is satisfied. By uniqueness of the process $H_k$, it follows that $H_0 = H_k$. Since the limit is the same for any subsequence $\{Z_{n_l}\}$, we conclude that the sequence $\{Z_n\}$ converges weakly to

$$Z_k = \left(H_k, \ldots, H_k^{(2k-2)}, Y_k, \ldots, Y_k^{(k-2)}, H_k^{(2k-1)}, Y_k^{(k-1)}\right)$$

and in particular $Z_n(0) \to_d Z_k(0)$ and $(\tilde{H}_k^l)^{(j)}(0) \to_d H_k^{(j)}(0)$ for $j = 0, \ldots, 2k-1$.

**Proof of Theorem 2.2.** For the direct problems, we apply Lemma 2.7 at $t = 0$ together with the fact that for $j = 0, \ldots, k - 1$,

$$(\tilde{H}_n^l)^{k+j}(0) = c_j(g_0)n^{(k-j)/(2k+1)}(\tilde{g}_n(x_0) - g_0(x_0))$$

and

$$(\hat{H}_n^l)^{k+j}(0) = c_j(g_0)n^{(k-j)/(2k+1)}(\hat{g}_n(x_0) - g_0(x_0)) \to_0 0 \quad \text{as} \quad n \to \infty$$

which follow from the respective definitions of $\tilde{H}_n^l$ and $\hat{H}_n^l$, and also uniform consistency of the MLE and its derivatives (for $\hat{H}_n^l$). For the inverse problem, the claim follows from Lemma 2.7 and the inverse formula in Balabdaoui and Wellner (2004a), Lemma 2.2.  

\[\square\]
5. Appendix.

5.1. VC-subgraph proofs for sections 3.3 and 4. Let \( g_t(x) = (x-t)^{k-1}/(k-1)! \) be the function in conjecture 2.1 and in (3.22) of the proof of Lemma 2.5. The notation for its Hermite interpolant (at \( \bar{\tau} \)) in section 2.3 is \( f(t) \). Here we change notation slightly, renaming the interpolant to be \( f_t(x) \) (to match \( g_t(x) \)). Then

\[
(5.43) \quad f_t(x) = \sum_{i=-(2k-1)}^{2k-4} \alpha_i B_i(x) = \sum_{i=-(2k-1)}^{2k-4} \left( \sum_{j=1}^{4k-4} M^{ij} g_j(t) \right) B_i(x)
\]

where the \( \{B_i\} = \{B_i(x; \tau)\} \) are the B-splines as in the proof of lemma 2.3, \([M^{ij}] = M^{-1}\) with \( M = M(\tau) \) as in (2.13), and \( g = g(t, \tau) \) is the \( 4k-4 \times 1 \) vector given by

\[
\begin{align*}
g_1(t) &= g_t(\tau_0) = \frac{(\tau_0 - t)^{k-1}}{(k-1)!}, \\
g_2(t) &= g'_t(\tau_0) = \frac{(\tau_0 - t)^{k-2}}{(k-2)!}, \\
\vdots \\
g_{4k-5}(t) &= g_t(\tau_{2k-3}) = \frac{(\tau_{2k-3} - t)^{k-1}}{(k-1)!}, \\
g_{4k-4}(t) &= g'_t(\tau_{2k-3}) = \frac{(\tau_{2k-3} - t)^{k-2}}{(k-2)!}.
\end{align*}
\]

Thus we consider the classes of functions \( \mathcal{F}^{(1)}_{y_0, R} \) and \( \mathcal{F}^{(2)}_{y_0, R} \) given by

\[
(5.44) \quad \mathcal{F}^{(1)}_{y_0, R} = \left\{ f_t(x) = f_t(x, y) : x \in [y_0, y_{2k-3}], \quad x_0 - \delta \leq y_0 < y_1 < \cdots < y_{2k-3} \leq y_0 + R \right\}
\]

and

\[
(5.45) \quad \mathcal{F}^{(2)}_{y_0, R} = \left\{ f_{y_0, \ldots, y_k}(t) : y_0 < y_1 < \cdots < y_{k-1} \leq y_0 + R \right\}
\]

where \( f_{y_0, \ldots, y_{k-1}} \) is as defined in (4.30). Here the components of \( y = (y_0, \ldots, y_{2k-3}) \) play the role of the \( \tau \)'s.

**Proposition 5.1** For \( k \geq 2 \) the classes of functions \( \mathcal{F}^{(1)}_{y_0, R} \) and \( \mathcal{F}^{(2)}_{y_0, R} \) given in (5.44) and (5.45) are VC-subgraph classes of functions.
Proof. We want to show that the class of subgraphs

\[ D = \left\{ (t, c) \in \mathbb{R}^+ \times \mathbb{R} : c < f_t(x) \right\} : \]
\[ x \in [\tau_0, \tau_{2k-3}], x_0 - \delta \leq y_0 < y_1 < \cdots < y_{2k-3} \leq y_0 + R \]

is a VC class of sets in \( \mathbb{R}^+ \times \mathbb{R} \). If we show this, then the class of functions (5.44) is VC-subgraph. Alternatively, from van der Vaart and Wellner (1996), problem 11, page 152, it suffices to show that the “between graphs”

\[ D_1 = \left\{ (t, c) \in \mathbb{R}^+ \times \mathbb{R} : 0 \leq c \leq f_t(x) \text{ or } f_t(x) \leq c \leq 0 \right\} : \]
\[ x \in [y_0, y_{2k-3}], x_0 - \delta \leq y_0 < y_1 < \cdots < y_{2k-3} \leq y_0 + R \]

is a VC class of sets. Let

\[ D_{1,j} = \left\{ (t, c) \in \mathbb{R}^+ \times \mathbb{R} : 0 \leq c \leq f_t(x)1_{[y_{j-1},y_j]}(t) \text{ or } f_t(x)1_{[y_{j-1},y_j]}(t) \leq c \leq 0 \right\} : \]
\[ x \in [\tau_0, \tau_{2k-3}], x_0 - \delta \leq y_0 < y_1 < \cdots < y_{2k-3} \leq y_0 + R \]

for \( j = 1, \ldots, 2k - 3 \). Since \( t \mapsto f_t(x)1_{[y_{j-1},y_j]}(t) \) is a polynomial of degree at most \( k - 1 \) for each \( j = 1, \ldots, k \), the classes \( D_{1,j} \) are all VC classes. Also note that

\[ D_1 \subset D_{1,1} \sqcup \cdots \sqcup D_{1,2k-3} = D_{1:k} \]

By Dudley (1999), theorem 2.5.3, page 153, \( D_{1:k} \) is a VC class (or see van der Vaart and Wellner (1996), Lemma 2.6.1, part (iii), page 147). Hence \( D_1 \) is a VC class and \( F^{(1)} \) is a VC - subgraph class.

The proof for the class \( F^{(2)}_{y_0,R} \) is similar.  

5.2. The rate control lemma. The following proposition is a slight generalization of Lemma 4.1 of Kim and Pollard (1990), page 201. It is used repeatedly in sections 3 and 4, while an adaptation of the argument is used in the next subsection of the appendix.
Lemma 5.1 Let \( F \) be a collection of functions defined on \([x_0 - \delta, x_0 + \delta]\), with \( \delta > 0 \) small and let \( \alpha > 0 \). Suppose that for a fixed \( x \in [x_0 - \delta, x_0 + \delta] \) and \( R > 0 \) such that \([x, x + R] \subseteq [x_0 - \delta, x_0 + \delta]\), the collection

\[
\mathcal{F}_{x, R} = \{ f_{x,y} \equiv f_{1[x,y]}, \; f \in F, \; x \leq y \leq x + R \}
\]

admits an envelope \( F_{x, R} \) such that

\[
EF_{x, R}^2(X_1) \leq KR^{2d-1}, \quad R \leq R_0,
\]

for some \( d \geq 1/2 \) where \( K > 0 \) depends only on \( x_0, \delta, \) and \( \alpha \). Moreover, suppose that

\[
\sup_Q \int_0^1 \sqrt{\log N(\eta\|F_{x, R}\|_{Q, 2}, F_{x, R}, L_2(Q))} d\eta < \infty.
\]

(5.46)

Then, for each \( \epsilon > 0 \) there exist random variables \( M_n = M_n(\alpha) \) of order \( O_p(1) \) such that

\[
(\mathcal{G}_n - G_0)(f_{x,y}) \leq \epsilon |y - x|^{\alpha + d} + n^{-\frac{\alpha + d}{2\alpha + 1}} M_n \quad \text{for } |y - x| \leq R_0.
\]

(5.47)

Proof. By Van der Vaart and Wellner (1996), theorem 2.14.1, page 239, it follows that

\[
\left\{ \sup_{f_{x,y} \in \mathcal{F}_{x, R}} |(\mathcal{G}_n - G_0)(f_{x,R})| \right\}^2 \leq \frac{K}{n} EF_{x, R}^2(X_1) = O \left( n^{-1} R^{2d-1} \right)
\]

(5.48)

for some constant \( K > 0 \) depending only on \( x_0, \delta, \) and the entropy integral in (5.46) For any \( f_{x,y} \in \mathcal{F}_{x, R} \), we write

\[
(\mathbb{P}_n - P_0)(f_{x,y}) = (\mathcal{G}_n - G_0)(f_{x,y})
\]

and define \( M_n \) by

\[
M_n = \inf \left\{ D > 0 : \left| (\mathbb{P}_n - P_0)(f_{x,y}) \right| \leq \epsilon |y - x|^{\alpha + d} + n^{-\alpha + d/(2\alpha + 1)} D, \right. \]

for all \( f_{x,y} \in \mathcal{F}_{x, R} \)
and \( M_n = \infty \) if no \( D > 0 \) satisfies the required inequality. For \( 1 \leq j \leq \lfloor \frac{R}{\alpha + 1} \rfloor = j_n \), we have

\[
P(M_n > m) \\leq P \left( \left| (\mathbb{P}_n - P_0)(f_{x,y}) \right| > \epsilon(y - x)^{\alpha+d} + n^{-(\alpha+d)/(2\alpha+1)}m \text{ for some } f_{x,y} \in \mathcal{F}_{x,R} \right) \\leq \sum_{1 \leq j \leq j_n} P \left\{ n^{(\alpha+d)/(2\alpha+1)} \left| (\mathbb{P}_n - P_0)(f_{x,y}) \right| > \epsilon(j - 1)^{\alpha+d} + m \right\} \\
\text{for some } f_{x,y} \in \mathcal{F}_{x,R}, (j - 1)n^{-1/(2\alpha+1)} \leq y - x \leq jn^{-1/(2\alpha+1)} \}
\]

\[
\leq \sum_{1 \leq j \leq j_n} n^{2(\alpha+d)/(2\alpha+1)} \mathbb{E} \left\{ \sup_{y:0 \leq y - x < jn^{-1/(2\alpha+1)}} \left| (\mathbb{P}_n - P_0)(f_{x,y-x}) \right| \right\}^2 \left( \epsilon(j - 1)^{\alpha+d} + m \right)^2 \\
= \sum_{1 \leq j \leq j_n} n^{2(\alpha+d)/(2\alpha+1)} \mathbb{E} \left\{ \sup_{f_{x,y-x} \in \mathcal{F}_{x,jn^{-1/(2\alpha+1)}}} \left| (\mathbb{P}_n - P_0)(f_{x,y-x}) \right| \right\}^2 \left( \epsilon(j - 1)^{\alpha+d} + m \right)^2 \\
\leq C \sum_{1 \leq j \leq j_n} n^{2(\alpha+d)/(2\alpha+1)} n^{-1/(2\alpha+1)} \frac{\epsilon(j - 1)^{2d-1}}{(\epsilon(j - 1)^{\alpha+d} + m)^2} \\
= C \sum_{1 \leq j \leq j_n} \frac{\epsilon(j - 1)^{2d-1}}{(\epsilon(j - 1)^{\alpha+d} + m)^2} \leq C \sum_{j=1}^{\infty} \frac{\epsilon(j - 1)^{2d-1}}{(\epsilon(j - 1)^{\alpha+d} + m)^2} \downarrow 0
\]
as \( m \not\to \infty \) where \( C > 0 \) is a constant that depends only on \( x_0, \delta \), and \( \alpha \). Therefore it follows that (5.47) holds. \( \blacksquare \)

5.3. \textit{Gap proof for the MLE}. Here we prove that the distances between the knots of the MLE, \( \hat{g}_n \), are of the stochastic order \( n^{-1/(2k+1)} \) as the sample size \( n \to \infty \) just as in the case of the LS estimator of a \( k \)-monotone density. Recall that \( \hat{g}_n \) is the MLE if and only if

\[
\int_0^x \frac{(x - t)^{k-1}}{\hat{g}_n(t)} \, d\mathcal{G}_n(t) \left\{ \begin{array}{ll}
\leq \frac{x^k}{k}, & \text{for } x > 0 \\
= \frac{x^k}{k}, & \text{if } x \text{ is a jump point of } \hat{g}_n^{(k-1)}.
\end{array} \right.
\]
Letting $\hat{G}_n(t) = \int_0^t \hat{g}_n(s) ds$, this can be rewritten as

$$
\int_0^x (x-t)^{k-1} \frac{d(\hat{G}_n(t) - G_n(t))}{g_n(t)} \begin{cases} 
  \geq 0, & \text{for } x > 0 \\
  = 0, & \text{if } x \text{ is a jump point of } \hat{g}_n^{(k-1)}.
\end{cases}
$$

We will use (5.49) to derive the stochastic order of the distance between the knots. In the following, we consider $\tau_0 < \ldots < \tau_{2k-3}$ to be $(2k-2)$ distinct jump points of $\hat{g}_n^{(k-1)}$. We also use the notation

$$
\hat{H}_n(x) = \int_0^x (x-t)^{k-1} d\hat{G}_n(t), \quad \Psi_n(x) = \int_0^x (x-t)^{k-1} dG_n(t).
$$

For $x \in [\tau_0, \tau_{2k-3}]$, we can write

$$
\int_0^x (x-t)^{k-1} \frac{d(\hat{G}_n(t) - G_n(t))}{g_n(t)} = \frac{1}{g_0(\tau_0)} \int_0^x (x-t)^{k-1} d(\hat{G}_n(t) - G_n(t))
$$

$$
= \frac{1}{g_0(\tau_0)} \int_0^x (x-t)^{k-1} d(\hat{G}_n(t) - G_n(t)) + \int_0^{\tau_0} (x-t)^{k-1} \left( \frac{1}{\hat{g}_n(t)} - \frac{1}{g_0(\tau_0)} \right) d(\hat{G}_n(t) - G_n(t))
$$

$$
= \frac{\hat{H}_n(x) - \Psi_n(x)}{g_0(\tau_0)} + \int_0^{\tau_0} (x-t)^{k-1} \left( \frac{1}{\hat{g}_n(t)} - \frac{1}{g_0(\tau_0)} \right) d(\hat{G}_n(t) - G_n(t))
$$

$$
+ \int_{\tau_0}^x (x-t)^{k-1} \left( \frac{1}{\hat{g}_n(t)} - \frac{1}{g_0(\tau_0)} \right) d(\hat{G}_n(t) - G_0(t))
$$

$$
+ \int_{\tau_0}^x (x-t)^{k-1} \left( \frac{1}{g_0(t)} - \frac{1}{g_0(\tau_0)} \right) d(G_0(t) - G_n(t))
$$

$$
= \frac{1}{g_0(\tau_0)} (\hat{H}_n(x) - \Psi_n(x)) + p_n(x) - \hat{f}_n(x) - \Delta_n(x)
$$

where

$$
p_n(x) \equiv \int_0^{\tau_0} (x-t)^{k-1} \left( \frac{1}{\hat{g}_n(t)} - \frac{1}{g_0(\tau_0)} \right) d(\hat{G}_n(t) - G_n(t))
$$

is a polynomial of degree $k-1$ (and hence it will be “filtered out” later on as shown in Lemma 1 below), and where, for all $x \in [\tau_0, \tau_{2k-3}]$,

$$
\Delta_n(x) \equiv \int_{\tau_0}^x (x-t)^{k-1} \left( \frac{1}{\hat{g}_n(t)} - \frac{1}{g_0(\tau_0)} \right) d(G_n(t) - G_0(t)),
$$

$$
\hat{f}_n(x) \equiv - \int_{\tau_0}^x (x-t)^{k-1} \left( \frac{1}{\hat{g}_n(t)} - \frac{1}{g_0(\tau_0)} \right) d(\hat{G}_n(t) - G_0(t)).
$$
We denote again by $H$ the Hermite interpolation operator that associates to each differentiable function $f$ the unique spline $s \in S_{2k-1}(\tau_0, \tau_1, \ldots, \tau_{2k-4}, \tau_{2k-3})$ such that

$$s(\tau_j) = f(\tau_j) \text{ and } s'(\tau_j) = f'(\tau_j), \quad j = 0, \ldots, 2k-3.$$ 

The following lemma is the first step towards deriving the stochastic order of $\tau_{2k-3} - \tau_0$ for the MLE:

**Lemma 5.2** For all $x \in [\tau_0, \tau_{2k-3}]$, we have

$$H \left[ \mathbb{V}_n + g_0(\tau_0)(\Delta_n + \hat{f}_n) \right](x) \geq \mathbb{V}_n(x) + g_0(\tau_0)(\Delta_n(x) + \hat{f}_n(x)).$$

**Proof.** It follows from the characterization in (5.49) that $\hat{H}_n$, which is in $S_{2k-1}(\tau_0, \tau_1, \ldots, \tau_{2k-4}, \tau_{2k-3})$, satisfies

$$\hat{H}_n(\tau_j) = \mathbb{V}_n(\tau_j) + g_0(\tau_0)(-p_n(\tau_j) + \Delta_n(\tau_j) + \hat{f}_n(\tau_j))$$

and

$$\hat{H}_n'(\tau_j) = \mathbb{V}_n'(\tau_j) + g_0(\tau_0)(-p_n'(\tau_j) + \Delta_n'(\tau_j) + \hat{f}_n'(\tau_j)),$$

and by uniqueness of the solution it follows that

$$\hat{H}_n = H \left( \mathbb{V}_n + g_0(\tau_0)(-p_n + \Delta_n + \hat{f}_n) \right)$$

on $[\tau_0, \tau_{2k-3}]$. But we have $H p_n = p_n$, and hence using the identity obtained in (5.50) the inequality condition in (5.49) can be rewritten as

$$H \left( \mathbb{V}_n + g_0(\tau_0)(\hat{f}_n + \Delta_n) \right) \geq \mathbb{V}_n + g_0(\tau_0)(\hat{f}_n + \Delta_n)$$

on $[\tau_0, \tau_{2k-3}]$. 

$\blacksquare$
Using linearity of the operator $H$, the previous result can be rewritten as

$$[H Y_n - Y_n] + g_0(\tau_0) [H \hat{f}_n - \hat{f}_n] + g_0(\tau_0) [H \Delta_n - \Delta_n] \geq 0 \text{ on } [\tau_0, \tau_{2k-3}]$$

or

$$\mathcal{E}_1 + \mathcal{E}_2 + \mathcal{E}_3$$

(5.51) \quad = \quad [Y_n - H Y_n] + g_0(\tau_0) [\hat{f}_n - H \hat{f}_n] + g_0(\tau_0) [\Delta_n - H \Delta_n] \leq 0$$

on $[\tau_0, \tau_{2k-3}]$. The second step is to evaluate the interpolation error for each term. We denote by $E_i, i = 1, 2, 3$ these error terms.

Let $\bar{\tau} \in (\tau_0, \tau_{2k-3})$. From our study of the distance between the knots of the LSE, we know already that if our conjecture is true, there exists a constant $D_1 > 0$ (depending only on $k$) such that

$$D_1 |g_0^{(k)}(x_0)| (\tau_{2k-3} - \tau_0)^{2k} (1 + o_p(1)) + O_p(n^{-(2k)/(2k+1)}),$$

independently of $\bar{\tau}$. Thus the inequalities in (5.51) and (5.52) imply that

We can also rewrite the preceding inequality as

$$D_1 |g_0^{(k)}(x_0)| (\tau_{2k-3} - \tau_0)^{2k} (1 + o_p(1)) + O_p(n^{-(2k)/(2k+1)}) + g_0(\tau_0) \mathcal{E}_2(\bar{\tau}) + g_0(\tau_0) \mathcal{E}_3(\bar{\tau}) \leq 0$$

which yields, in turn,

$$\begin{align*}
D_1 |g_0^{(k)}(x_0)| (\tau_{2k-3} - \tau_0)^{2k} (1 + o_p(1)) & + O_p(n^{-(2k)/(2k+1)}) \\
& \leq g_0(\tau_0) \sup_{\bar{\tau} \in [\tau_0, \tau_{2k-3}]} |\mathcal{E}_2(\bar{\tau})| + g_0(\tau_0) \sup_{\bar{\tau} \in [\tau_0, \tau_{2k-3}]} |\mathcal{E}_3(\bar{\tau})|.
\end{align*}$$

(5.53)

We start by showing that

$$\sup_{\bar{\tau} \in [\tau_0, \tau_{2k-3}]} |\mathcal{E}_3(\bar{\tau})| = o((\tau_{2k-3} - \tau_0)^{2k} + o_p(n^{-(2k)/(2k+1)}).$$

We have

$$|\mathcal{E}_3(\bar{\tau})|$$
In other words, one can view $E_3(\tau)$ in the following way: For a fixed $t \in [\tau_0, \tau_{2k-3}]$, we compute the value at the point $\bar{\tau}$ of the Hermite interpolation error for interpolating the function

$$x \mapsto (x - t)^{k-1}1_{[\tau_0, \tau]}(t) \left( \frac{1}{g_n(t)} - \frac{1}{g_0(\tau_0)} \right)$$

or, since $1_{[\tau_0, x]}(t) = 1_{[x \geq t]}$ since $t \geq \tau_0$,

$$x \mapsto (x - t)^{k-1}1_{[x \geq t]}(t) \left( \frac{1}{g_n(t)} - \frac{1}{g_0(\tau_0)} \right)$$

(5.55)

This yields a function of $t$, which is then integrated with respect to $(G_n - G_0)$.

Let us denote by

$$f_{\tau_0, \ldots, \tau_{2k-3}, \lambda, g_n}(t)$$

the function that assigns to each $t \in [\tau_0, \tau_{2k-4}]$ the Hermite interpolation error at the point $\bar{\tau}$ for interpolating the function defined in (5.55), where $\bar{\lambda} = (\bar{\tau} - \tau_0)/(\tau_{2k-3} - \tau_0)$.

Let $\epsilon > 0$, and $\delta > 0$ such that $[\tau_0, \tau_{2k-4}] \subset [x_0 - \delta, x_0 + \delta]$. By uniform consistency of the derivative of $g_n^{(j)}$, $j = 0, \ldots, k - 2$, for every $\gamma > 0$ there exists an $N = N_\gamma \in \mathbb{N}$ such that for $n > N$, the probability of the event

$$\{ \omega: \max_{0 \leq j \leq k-2} \sup_{t \in [x_0 - \delta, x_0 + \delta]} \left| \left( \frac{1}{g_n(\omega, t)} - \frac{1}{g_0(\tau_0)} \right)^{(j)} \right| \leq \gamma \}$$

(5.56)

is greater than $1 - \epsilon$. In what follows, we consider the case where the previous event occurs.
Now fix $y_0 \in [x_0 - \delta, x_0 + \delta - R]$. Consider the collection
\[
\mathcal{F}_{y_0, R, \gamma} = \left\{ f_{y_0, y_1, \cdots, y_{2k-3}, \lambda, s} : y_0 \leq y_1 \leq \cdots \leq y_{2k-4} \leq y_{2k-3} \leq y_0 + R, \right.
\]
\[
\lambda \in [0, 1], \text{ and } s \in C_{\gamma}^{k-2}[x_0 - \delta, x_0 + \delta] \left\} \right.
\]
where $C_{\gamma}^{k-2}[x_0 - \delta, x_0 + \delta]$ is the set of functions on $[x_0 - \delta, x_0 + \delta]$ whose $j$-th derivative, $j = 0, \cdots, k - 2$, is uniformly bounded by $\gamma$. Explicitly, a function in the previous collection can be written as
\[
f_{y_0, y_1, \cdots, y_{2k-3}, \lambda, s}(t) = \left\{ (\lambda y_0 + (1 - \lambda)y_{2k-3} - t)^{k-1}_+ - [H(\cdot - t)^{k-1}_+] (\lambda y_0 + (1 - \lambda)y_{2k-3}) \right\} \cdot s(t),
\]
and hence if we denote by $\mathcal{F}_{y_0, R}$, the collection of functions appearing in the first term on the right side of the previous display, we have
\[
\mathcal{F}_{y_0, R, \gamma} \subset \mathcal{F}_{y_0, R} \cdot C_{\gamma}^{k-2}[x_0 - \delta, x_0 + \delta].
\]
Fix $\eta > 0$, and let $Q$ be a probability measure on $(0, \infty)$. Now using the same arguments as in the proof of Proposition 6.1, the collection $\mathcal{F}_{y_0, R}$ is VC, and we can find $D_1 = D_1(\delta, k) < \infty$ such that
\[
\log N(\eta \| F_{y_0, R} \|_{Q, 2}, \mathcal{F}_{y_0, R}, L_2(Q)) \leq D_1 \log \frac{1}{\eta}.
\]
On the other hand, by Theorem 2.7.1 of Van der Vaart and Wellner (1996), page 155, there exists $D_2 = D_2(\delta, k) < \infty$
\[
\log N(\eta \gamma, C_{\gamma}^{k-2}[x_0 - \delta, x_0 + \delta], \| \cdot \|_{\infty}) \leq D_2 \left( \frac{1}{\eta} \right)^{\frac{1}{k-2}},
\]
where the constant $D_2$ depends on $k$, and $\delta$, but not on $x_0$. Note that $\mathcal{F} \equiv \mathcal{F}_{y_0, R}$ has bounded envelope function $F \equiv F_{y_0, R}$. Thus if $\{f_j\}$ is an $\eta \| F \|_{y_0, R} - \text{net}$ with respect to $L_2(Q)$ for $\mathcal{F} \equiv \mathcal{F}_{y_0, R}$ and $\{g_{j'}\}$ is an $\eta \gamma - \text{net}$ with respect to $\| \cdot \|_{\infty}$ for $\mathcal{G} \equiv C_{\gamma}^{k-2}[x_0 - \delta, x_0 + \delta]$, then $\{f_j \cdot g_{j'}\}$ is a $2\eta \gamma \| F \|_{Q, 2} - \text{net}$ for $\mathcal{F} \cdot \mathcal{G}$ with respect to $L_2(Q)$: for $f, g$ with $\| f - f_j \|_{Q, 2} \leq
\[ \epsilon \|F\|_{Q,2} \text{ and } \|g - g_j'\|_\infty \leq \epsilon \gamma, \]
\[ \|f \cdot g - f_j g_j'\|_{Q,2} \leq \gamma \|f - f_j\|_{Q,2} + \|F\|_{Q,2} \|g - g_j'\|_\infty \]
\[ \leq \gamma \eta \|F\|_{Q,2} + \|F\|_{Q,2} \eta \gamma = 2\eta \gamma \|F\|_{Q,2}. \]

It follows that
\[ N(2\eta \gamma \|F\|_{Q,2}, F_{y_0, R, \gamma, L_2(Q)}) \leq N(\eta \|F\|_{Q,2}, F_{y_0, R, L_2(Q)}) \cdot N(\eta \gamma, C_k^{k-2}[x_0 - \delta, x_0 + \delta], L_2(Q)). \]
(5.57)

By (5.57) and dominance of the second entropy bound as \( \eta \downarrow 0 \), we conclude that
\[ \log N(\eta \gamma \|F\|_{Q,2}, F_{y_0, R, \gamma, L_2(Q)}) \leq K \left( \frac{1}{\eta} \right) \frac{1}{k-1}. \]

where \( K \) depends on \( k \) and \( \delta \) (but not on \( R \) or \( Q \)). On the other hand, it follows from the error boundedness Conjecture 3.1 (also see Balabdaoui and Wellner (2005)) that \( F_{x, R, \gamma} \) admits the function
\[ F_{y_0, R, \gamma}(t) = C \gamma R^{k-11}[y_0, y_0 + R](t) \]
as an envelope, where \( C > 0 \) depends only on \( k \). Now we can find a constant \( D > 0 \) depending only on \( \eta \) and \( g_0 \) and such that \( 0 < \sup_{t \in [x_0 - \delta, x_0 + \delta]} g_0(t) \leq D \). We can write
\[ E F_{y_0, R, \gamma}^2(X_1) = C^2 \gamma^2 R^{2(k-1)} \int_{y_0}^{y_0 + R} g_0(t) dt \leq C^2 D^2 \gamma^2 R^{2k-1}, \]
and hence by van der Vaart and Wellner (1996), Theorem 2.14.2, page 240, there exists a constant \( K' \) depending only on \( x_0 \) and \( \delta \) such that
\[ E \left\{ \left( \sup_{f_{y_0, y_1, \ldots, y_{2k-3}, \lambda, s} \in F_{y_0, R, \gamma}} \|G_n - G_0\|_2 \right)^2 \right\} \leq \frac{K'}{n} E F_{y_0, R, \gamma}^2(X_1) = K'' n^{-1} \gamma^2 R^{2k-1}. \]
Using notation similar to that of subsection 5.2 and 4, we define $M_n$ by

$$
M_n = \inf \left\{ m > 0 : \left| (P_n - P_0)(f_{y_0, y_1, \ldots, y_{2k-3}, \lambda, s}) \right| \leq \epsilon (y_{2k-3} - y_0)^{2k} + n^{-2k/(2k+1)}m, \text{ for all } f_{y_0, y_1, \ldots, y_{2k-3}, \lambda, s} \in \mathcal{F}_{y_0, R, \gamma} \right\}
$$

and $M_n = \infty$ if no $m > 0$ satisfies the required inequality. For $1 \leq j \leq j_n = \lceil Rn^{1/(2k+1)} \rceil$, we have

$$
P(M_n > d) \leq P\left( \left| (P_n - P_0)(f_{y_0, y_1, \ldots, y_{2k-3}, \lambda, s}) \right| > \epsilon (y_{2k-3} - y_0)^{2k} + n^{-2k/(2k+1)}d \right)
$$

for some $f_{y_0, y_1, \ldots, y_{2k-3}, \lambda, s} \in \mathcal{F}_{y_0, R, \gamma}$.

$$
= \sum_{1 \leq j \leq j_n} P\left( n^{2k/(2k+1)} \left| (P_n - P_0)(f_{y_0, y_1, \ldots, y_{2k-3}, \lambda, s}) \right| > \epsilon (j - 1)^{2k} + d \right)
$$

for some $f_{y_0, y_1, \ldots, y_{2k-3}, \lambda, s} \in \mathcal{F}_{y_0, R, \gamma}$,

$$
(j - 1)n^{-1/(2k+1)} \leq y_{2k-3} - y_0 \leq jn^{-1/(2k+1)} \right\}
$$

$$
\leq \sum_{1 \leq j \leq j_n} n^{4k + 1} \sup_{f \in \mathcal{F}_{y_0, jn^{-1/(2k+1)}, \gamma}} \left| (P_n - P_0)(f_{y_0, y_1, \ldots, y_{2k-3}, \lambda, s}) \right|^2
$$

$$
\leq K'' \sum_{1 \leq j \leq j_n} n^{4k + 1} n^{-1} \gamma^2 n^{-2k/(2k+1)} j^{2k-1} \epsilon (j - 1)^{2k} + d \right|^2
$$

$$
= K'' \gamma^2 \sum_{1 \leq j \leq j_n} \frac{j^{2k-1}}{(\epsilon (j - 1)^{2k} + d)^2}
$$

$$
\leq K'' \gamma^2 \sum_{j=1}^{\infty} \frac{j^{2k-1}}{(\epsilon (j - 1)^{2k} + d)^2}
$$

and the latter can be be made arbitrarily small by choice of $\gamma$ for each fixed $d > 0$: in the definition of $\mathcal{F}_{y_0, R, \gamma}$, $\gamma$ can be chosen to be as small as desired by uniform consistency of $\hat{g}_n^{(j)}$, $j = 0, \ldots, k - 2$; recall the definition of the event $J_{x_0, \delta, \gamma}$ given in (5.56). Therefore, $M_n = o_p(1)$ implying that

$$
\sup_{\lambda \in [0,1]} \left| (P_n - P_0)(f_{\tau_0, \tau_1, \ldots, \tau_{2k-3}, \lambda, \delta_n}) \right| = o_p(\tau_{2k-3} - \tau_0)^{2k} + o_p(n^{-2k/(2k+1)})
$$
or equivalently

\[
\sup_{\tau \in [\tau_0, \tau_{2k-3}]} |\mathcal{E}_3(\tau)| = o_p(\tau_{2k-3} - \tau_0)^{2k} + o_p(n^{-2k/(2k+1)}).
\]

Finally, we focus on the last error term, \(\mathcal{E}_2\). Recall that the corresponding interpolated function is given by

\[
\tilde{f}_n(\tau) = -\int_{\tau_0}^\tau (\tau - t)^{k-1} \frac{1}{\hat{g}_n(t)} (\hat{g}_n(t) - g_0(\tau_0))(\hat{g}_n(t) - g_0(t)) dt
\]

for all \(\tau \in [\tau_0, \tau_{2k-3}]\). Note that the function is \((2k-1)\)-times differentiable on \([\tau_0, \tau_{2k-3}]\), and we have

\[
\|H\tilde{f}_n - \tilde{f}_n\|_\infty \leq \frac{1}{(2k-1)!} \int_{\tau_0}^{\tau_{2k-3}} \|H[(t-\cdot)^{2k-1}] - (t-\cdot)^{2k-1}\|_\infty |d\tilde{f}_n^{(2k-1)}(t)|
\]

\[
\leq d_k(\tau_{2k-3} - \tau_0)^{2k-1} \int_{\tau_0}^{\tau_{2k-3}} |d\tilde{f}_n^{(2k-1)}(t)|,
\]

for some \(d_k > 0\). For a proof of the latter bound, see e.g. Lemma 2.1 of Balabdaoui and Wellner (2005). On the other hand, we have

\[
\tilde{f}_n^{(2k-1)}(t) = \left[ \left( \hat{g}_n(t) - g_0(t) \right) \left( \frac{\hat{g}_n(t) - g_0(\tau_0)}{\hat{g}_n(t)} \right) \right]^{(k-1)}
\]

\[
= \sum_{j=0}^{k-1} \binom{k-1}{j} \left( \hat{g}_n^{(j)}(t) - g_0^{(j)}(t) \right) \left( \frac{\hat{g}_n(t) - g_0(\tau_0)}{\hat{g}_n(t)} \right)^{(k-1-j)},
\]

and hence

\[
d\tilde{f}_n^{(2k-1)}(t)
\]

\[
= (\hat{g}_n(t) - g_0(t)) d \left[ \left( \frac{\hat{g}_n(t) - g_0(\tau_0)}{\hat{g}_n(t)} \right)^{(k-1)} \right]
\]

\[
+ d \left( \hat{g}_n^{(k-1)}(t) - g_0^{(k-1)}(t) \right) \left( \frac{\hat{g}_n(t) - g_0(\tau_0)}{\hat{g}_n(t)} \right)
\]

\[
+ \sum_{j=1}^{k-2} \binom{k-1}{j} \left( \hat{g}_n^{(j+1)}(t) - g_0^{(j+1)}(t) \right) \left( \frac{\hat{g}_n(t) - g_0(\tau_0)}{\hat{g}_n(t)} \right)^{(k-1-j)} dt
\]
The last two functions, \( dh_3 \) and \( dh_4 \) are easier to handle, since we can see that uniform consistency of the derivatives of the MLE implies that
\[
\limsup_{n \to \infty} \sup_{t \in [\tau_0, \tau_{2k-3}]} |h_3'(t)| = \limsup_{n \to \infty} |h_4'(t)| = o_p(1),
\]
and hence
\[
(\tau_{2k-3} - \tau_0)^{2k-1} \int_{\tau_0}^{\tau_{2k-3}} |h_3'(t) + h_4'(t)| dt = o_p((\tau_{2k-3} - \tau_0)^{2k}).
\]

As for \( dh_1 \) and \( dh_2 \), we need the following lemma:

**Lemma 5.3** For any \( \epsilon > 0 \), there exists \( K > 0 \) (depending on \( k \)) such that for \( j = 1, \ldots, 2k-3 \) the event
\[
0 < (-1)^{k-1}(\hat{g}_n^{(k-1)}(\tau_j) - \hat{g}_n^{(k-1)}(\tau_{j-1})) < K (\tau_j - \tau_{j-1})
\]
occurs with probability greater than \( 1 - \epsilon \).

**Proof.** A picture is sufficient to prove the lemma, but more formally we have for \( x \in [x_0 - \delta, x_0 + \delta] \) for small \( \delta > 0 \)
\[
\frac{\hat{g}_n^{(k-2)}(x - h) - \hat{g}_n^{(k-2)}(x)}{h} \leq \hat{g}_n^{(k-1)}(x-) \leq \hat{g}_n^{(k-1)}(x+) \leq \frac{\hat{g}_n^{(k-2)}(x + h) - \hat{g}_n^{(k-2)}(x)}{h}
\]
(we assume here that \( k \) is even). We denote by \( \Delta \hat{g}_n^{(k-1)}(x) \) the height of the jump of \( \hat{g}_n^{(k-1)} \) at the point \( x \); i.e., \( \Delta \hat{g}_n^{(k-1)}(x) = \hat{g}_n^{(k-1)}(x+) - \hat{g}_n^{(k-1)}(x-) \), and by \( \Delta x \) the value of the corresponding gap (if \( x = \tau_j \), then \( \Delta x = \tau_j - \tau_{j-1} \)). The inequality in (5.58) implies that for all \( 0 < h < \Delta x \), we have
\[
0 \leq \limsup_{n \to \infty} \frac{\Delta \hat{g}_n^{(k-1)}(x)}{\Delta x} \leq \frac{1}{h} \left\{ \frac{g_0^{(k-2)}(x + h) - g_0^{(k-2)}(x)}{h} - \frac{g_0^{(k-2)}(x - h) - g_0^{(k-2)}(x)}{-h} \right\}.
\]
On the other hand, we know from our working assumptions that \( g^{(k-2)}_0 \) is twice continuously differentiable in the neighborhood of \( x_0 \). Therefore, using Taylor expansion, we have
\[
g^{(k-2)}_0(x + h) = g^{(k-2)}_0(x) + hg^{(k-1)}_0(x) + \frac{1}{2}h^2g^{(k)}_0(x) + o(h^2)
\]
and
\[
g^{(k-2)}_0(x - h) = g^{(k-2)}_0(x) - hg^{(k-1)}_0(x) + \frac{1}{2}h^2g^{(k)}_0(x) + o(h^2)
\]
and hence
\[
\frac{1}{h}\left\{\frac{g^{(k-2)}(x + h) - g^{(k-2)}(x)}{h} - \frac{g^{(k-2)}(x - h) - g^{(k-2)}(x)}{h}\right\} = g^{(k)}(x) + o(1)
\]
where \( K \) can be taken e.g. to be equal to \( 2 \sup_{t \in [x_0 - \delta, x_0 + \delta]} |g^{(k)}_0(t)| \). It follows that for \( n \) large enough and for all \( j \in 1, \ldots, 2k - 3 \),
\[
0 < \hat{g}^{(k-1)}_n(\tau_j) - \hat{g}^{(k-1)}_n(\tau_{j-1}) \leq K (\tau_j - \tau_{j-1})
\]
with large probability. \( \blacksquare \)

Now we go back to \( h_1 \) and \( h_2 \) and the corresponding error terms, and we start with \( h_2 \). Recall that
\[
\int_{\tau_0}^{\tau_{2k-3}} |d\hat{h}_2(t)| = \int_{\tau_0}^{\tau_{2k-3}} \left| d \left( \hat{g}^{(k-1)}_n(t) - g^{(k-1)}_0(t) \right) \left( \frac{\hat{g}_n(t) - g_0(\tau_0)}{\hat{g}_n(t)} \right) \right| \\
\leq \int_{\tau_0}^{\tau_{2k-3}} \left| d\hat{g}^{(k-1)}_n(t) \left( \frac{\hat{g}_n(t) - g_0(\tau_0)}{\hat{g}_n(t)} \right) \right| \\
+ \int_{\tau_0}^{\tau_{2k-3}} \left| dg^{(k-1)}_0(t) \left( \frac{\hat{g}_n(t) - g_0(\tau_0)}{\hat{g}_n(t)} \right) \right|.
\]
The second term is \( o_p((\tau_{2k-3} - \tau_0)) \) since \( dg^{(k-1)}_0(t) = g^{(k)}_0(t)dt \) (we apply the same argument used for \( dh_3 \) and \( dh_4 \)). As for the first term, we have
\[
\int_{\tau_0}^{\tau_{2k-3}} \left| d\hat{g}^{(k-1)}_n(t) \left( \frac{\hat{g}_n(t) - g_0(\tau_0)}{\hat{g}_n(t)} \right) \right| 
\]
\[
\begin{align*}
&= \sum_{j=1}^{2k-3} (\hat{g}_n^{(k-1)}(\tau_j) - \hat{g}_n^{(k-1)}(\tau_{j-1})) \left| \frac{\hat{g}_n(\tau_j) - g_0(\tau_0)}{\hat{g}_n(\tau_j)} \right| \\
&\leq D \sum_{j=1}^{2k-3} (\tau_j - \tau_{j-1}) \left| \frac{\hat{g}_n(\tau_j) - g_0(\tau_0)}{\hat{g}_n(\tau_j)} \right| \\
&\leq D(\tau_{2k-3} - \tau_0) \sum_{j=1}^{2k-3} \left| \frac{\hat{g}_n(\tau_j) - g_0(\tau_0)}{\hat{g}_n(\tau_j)} \right| \\
&= o_p((\tau_{2k-3} - \tau_0))
\end{align*}
\]

by uniform consistency of the MLE and continuity of \(g_0\) which imply that \(\hat{g}_n(\tau_j) - g_0(\tau_0) = o_p(1)\) for \(j = 1, \ldots, 2k - 3\). Similar arguments can be used for \(h_1\). We conclude that the associated error term is of the order

\[o_p((\tau_{2k-3} - \tau_0)^{2k}),\]

or using our notation above

\[\sup_{\tau \in [\tau_0, \tau_{2k-3}]} |\mathcal{E}_3(\tau)| = o_p((\tau_{2k-3} - \tau_0)^{2k}).\]

Therefore, in view of (5.53) and the previous results, we obtain

\[D_1|g_0^{(k)}(x_0)|(\tau_{2k-3} - \tau_0)^{2k}(1 + o_p(1)) + O_p(n^{-2k/(2k+1)}) \leq 0\]

which implies that

\[\tau_{2k-3} - \tau_0 = O_p(n^{-1/(2k+1)}).\]

5.4. **Reduction of the B-spline formula when \(k = 2\).** In this appendix we specialize Lemma 3.3 to the case \(k = 2\) and show that we recover the explicit cubic spline formulas of GJW (2001a), formula (2.18), page 1631.

When \(k = 2\), we have knots \(\tau_j, j = -3, -2, -1, 0, 1, 2, 3, 4\), and Lemma 3.3 implies that

\[\hat{H}_n(t) = \sum_{i=-3}^{0} \alpha_i B_i^3(t) \equiv \sum_{i=-3}^{0} \alpha_i B_i(t),\]

(5.59)
where

\[(5.60) \quad B_i(t) = \sum_{j=i}^{i+4} a_{j,i}(t - x_j)_+^3.\]

The \(\alpha\)'s satisfy

\[
B \alpha \equiv \begin{pmatrix} B_{-3}(\tau_0) & B_{-2}(\tau_0) & B_{-1}(\tau_0) & B_0(\tau_0) \\ B'_{-3}(\tau_0) & B'_{-2}(\tau_0) & B'_{-1}(\tau_0) & B'_0(\tau_0) \\ B_{-3}(\tau_1) & B_{-2}(\tau_1) & B_{-1}(\tau_1) & B_0(\tau_1) \\ B'_{-3}(\tau_1) & B'_{-2}(\tau_1) & B'_{-1}(\tau_1) & B'_0(\tau_1) \end{pmatrix} \begin{pmatrix} \alpha \end{pmatrix} = \begin{pmatrix} Y_n(\tau_0) \\ Y'_n(\tau_0) \\ Y_n(\tau_1) \\ Y'_n(\tau_1) \end{pmatrix} \equiv \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{pmatrix}
\]

We need to compute the matrix on the left side. To this end:

\[
B_0(\tau_0) = \sum_{j=0}^{4} a_{j,0}(\tau_0 - x_j)_+^3 = 0, \quad B'_0(\tau_0) = \sum_{j=0}^{4} 3a_{j,0}(\tau_0 - x_j)_+^2 = 0
\]

since none of the terms contribute (due to the positive part signs), and, on the other hand,

\[
B_{-3}(\tau_1) = \sum_{j=-3}^{1} a_{j,-3}(\tau_1 - x_j)_+^3 = 0, \quad B'_{-3}(\tau_1) = \sum_{j=-3}^{1} 3a_{j,-3}(\tau_1 - x_j)_+^2 = 0
\]

since the function \(B_{-3}(t) = 0\) for \(t \geq \tau_1\) (and this occurs also for the first derivative). Furthermore,

\[
B_0(\tau_1) = \sum_{j=0}^{4} a_{j,0}(\tau_1 - x_j)_+^3 = a_{0,0}(\tau_1 - \tau_0)^3,
\]

\[
B'_0(\tau_1) = \sum_{j=0}^{4} 3a_{j,0}(\tau_1 - x_j)_+^2 = 3a_{0,0}(\tau_1 - \tau_0)^2.
\]

Working our way left in the first two rows,

\[
B_{-1}(\tau_0) = \sum_{j=-1}^{3} a_{j,-1}(\tau_0 - x_j)_+^3 = a_{-1,-1}(\tau_0 - \tau_{-1})^3,
\]

\[
B'_{-1}(\tau_0) = \sum_{j=-1}^{3} 3a_{j,-1}(\tau_0 - x_j)_+^2 = 3a_{-1,-1}(\tau_0 - \tau_{-1})^2.
\]
\[ B_{-2}(\tau_0) = \sum_{j=-2}^{2} a_{j,-2}(\tau_0 - x_j)_+^3 = a_{-2,-2}(\tau_0 - \tau_{-2})^3 + a_{-1,-2}(\tau_0 - \tau_{-1})^3, \]
\[ B'_{-2}(\tau_0) = \sum_{j=-2}^{2} 3a_{j,-2}(\tau_0 - x_j)_+^2 = 3a_{-2,-2}(\tau_0 - \tau_{-2})^2 + 3a_{-1,-2}(\tau_0 - \tau_{-1})^2, \]
\[ B_{-3}(\tau_0) = \sum_{j=-3}^{1} a_{j,-3}(\tau_0 - x_j)_+^3 = a_{-3,-3}(\tau_0 - \tau_{-3})^3 + a_{-2,-3}(\tau_0 - \tau_{-2})^3 + a_{-1,-3}(\tau_0 - \tau_{-1})^3, \]
\[ B'_{-3}(\tau_0) = \sum_{j=-3}^{1} 3a_{j,-3}(\tau_0 - x_j)_+^2 = 3a_{-3,-3}(\tau_0 - \tau_{-3})^2 + 3a_{-2,-3}(\tau_0 - \tau_{-2})^2 + 3a_{-1,-3}(\tau_0 - \tau_{-1})^2. \]

The remaining pieces are given by
\[ B_{-1}(\tau_1) = \sum_{j=-1}^{3} a_{j,-1}(\tau_1 - x_j)_+^3 = a_{-1,-1}(\tau_1 - \tau_{-1})^3 + a_{0,-1}(\tau_1 - \tau_0)^3, \]
\[ B'_{-1}(\tau_1) = \sum_{j=-1}^{3} 3a_{j,-1}(\tau_1 - x_j)_+^2 = 3a_{-1,-1}(\tau_1 - \tau_{-1})^2 + 3a_{0,-1}(\tau_1 - \tau_0)^2, \]
\[ B_{-2}(\tau_1) = \sum_{j=-2}^{2} a_{j,-2}(\tau_1 - x_j)_+^3 = a_{-2,-2}(\tau_1 - \tau_{-2})^3 + a_{-1,-2}(\tau_1 - \tau_{-1})^3 + a_{0,-2}(\tau_1 - \tau_0)^3, \]
\[ B'_{-2}(\tau_1) = \sum_{j=-2}^{2} 3a_{j,-2}(\tau_1 - x_j)_+^2 = 3a_{-2,-2}(\tau_1 - \tau_{-2})^2 + 3a_{-1,-2}(\tau_1 - \tau_{-1})^2 + 3a_{0,-2}(\tau_1 - \tau_0)^2. \]

Since the formula for $\tilde{H}_n$ is invariant under choice of \( \{\tau_j, j = -3, -2, -1\} \), we can take them to have the same spacing from each other as $\tau_0$ and $\tau_1$, namely $\tau_1 - \tau_0$. Thus
\[ \tau_0 - \tau_{-3} = 3(\tau_1 - \tau_0), \quad \tau_0 - \tau_{-2} = 2(\tau_1 - \tau_0), \quad \tau_0 - \tau_{-1} = (\tau_1 - \tau_0). \]
while
\[ \tau_1 - \tau_{-2} = 3(\tau_1 - \tau_0), \quad \tau_1 - \tau_{-1} = 2(\tau_1 - \tau_0). \]

By elementary calculations using Lemma 3.2, the \( a_{i,j} \)'s are given by
\[
\begin{pmatrix}
a_{-3,3} & a_{-2,3} & a_{-1,3} \\
a_{-2,2} & a_{-1,2} & a_{0,-2} \\
a_{-1,1} & a_{0,-1} \\
a_{0,0}
\end{pmatrix} =
\begin{pmatrix}
\frac{1}{6} & -\frac{2}{3} & 1 \\
1/6 & -\frac{2}{3} & 1 \\
1/6 & -\frac{2}{3} & 1
\end{pmatrix}.
\]

Putting these facts together we find that
\[
B = \begin{pmatrix}
\frac{1}{6}d^3 & \frac{2}{3}d^3 & \frac{1}{6}d^3 & 0 \\
-\frac{1}{2}d^2 & 0 & \frac{1}{2}d^2 & 0 \\
0 & \frac{1}{6}d^3 & \frac{2}{3}d^3 & \frac{1}{6}d^3 \\
0 & -\frac{1}{2}d^2 & 0 & \frac{1}{2}d^2
\end{pmatrix} = \frac{1}{6} \begin{pmatrix}
d^3 & 4d^3 & d^3 & 0 \\
-3d^2 & 0 & 3d^2 & 0 \\
0 & d^3 & 4d^3 & d^3 \\
0 & -3d^2 & 0 & 3d^2
\end{pmatrix}.
\]

where \( d \equiv \tau_1 - \tau_0 \). We note that
\[
A \equiv B^{-1} = \frac{1}{3d^3} \begin{pmatrix}
-3 & -7d & 6 & -2d \\
3 & 2d & -3 & d \\
-3 & -d & 6 & -2d \\
6 & 2d & -3 & 7d
\end{pmatrix}.
\]

Solving for \( \alpha = (\alpha_{-3}, \alpha_{-2}, \alpha_{-1}, \alpha_0) \) yields
\[
\alpha = \frac{1}{3d^3} \begin{pmatrix}
-3\Upsilon_n(\tau_0) - 7d\Upsilon'_n(\tau_0) + 6\Upsilon_n(\tau_1) - 2d\Upsilon'_n(\tau_1) \\
6\Upsilon_n(\tau_0) + 2d\Upsilon'_n(\tau_0) - 3\Upsilon_n(\tau_1) + d\Upsilon'_n(\tau_1) \\
-3\Upsilon_n(\tau_0) - d\Upsilon'_n(\tau_0) + 6\Upsilon_n(\tau_1) - 2d\Upsilon'_n(\tau_1) \\
6\Upsilon_n(\tau_0) + 2d\Upsilon'_n(\tau_0) - 3\Upsilon_n(\tau_1) + 7d\Upsilon'_n(\tau_1)
\end{pmatrix}.
\]
Using this in (5.59) yields a formula for $\tilde{H}_n$ which agrees with the (finite-sample analogue) of (2.18) on page 1631 of GJW (2001a).

**Proposition.** In the case $k = 2$,

$$
\tilde{H}_n(x) = \frac{1}{3d^3} \left( \begin{array}{c}
-3\overline{Y}_n(\tau_0) - 7\overline{Y}'_n(\tau_0) + 6\overline{Y}_n(\tau_1) - 2\overline{Y}'_n(\tau_1) \\
6\overline{Y}_n(\tau_0) + 2d\overline{Y}'_n(\tau_0) - 3\overline{Y}_n(\tau_1) + d\overline{Y}'_n(\tau_1) \\
-3\overline{Y}_n(\tau_0) - d\overline{Y}'_n(\tau_0) + 6\overline{Y}_n(\tau_1) - 2d\overline{Y}'_n(\tau_1) \\
6\overline{Y}_n(\tau_0) + 2d\overline{Y}'_n(\tau_0) - 3\overline{Y}_n(\tau_1) + 7d\overline{Y}'_n(\tau_1)
\end{array} \right)^T \left( \begin{array}{c}
B_{-3}(x) \\
B_{-2}(x) \\
B_{-1}(x) \\
B_0(x)
\end{array} \right)
$$

$$
= \frac{\overline{Y}_n(\tau_1)(x - \tau_0) + \overline{Y}_n(\tau_0)(\tau_1 - x)}{\Delta \tau} - \frac{1}{2} \left\{ \frac{\Delta \overline{Y}'_n}{\Delta \tau} + 4\left( \frac{\overline{Y}_n \Delta \tau - \Delta \overline{Y}_n}{(\Delta \tau)^3} \right)(x - \tau_0)(\tau_1 - x) \right\},
$$

where $\Delta \overline{Y}'_n = \overline{Y}'_n(\tau_1) - \overline{Y}'_n(\tau_0)$, $\overline{Y}'_n = (\overline{Y}_n(\tau_0) + \overline{Y}_n(\tau_1))/2$, and $\Delta \overline{Y}_n = \overline{Y}_n(\tau_1) - \overline{Y}_n(\tau_0)$.

$$
\left( \begin{array}{c}
B_{-3}(x) \\
B_{-2}(x) \\
B_{-1}(x) \\
B_0(x)
\end{array} \right) = \frac{1}{6} \left( \begin{array}{c}
(1/6)(\tau_1 - x)^3 \\
(1/6)(2\tau_1 - \tau_0 - x)^3 - (2/3)(\tau_1 - x)^3 \\
(1/6)(x - 2\tau_0 + \tau_1)^3 - (2/3)(x - \tau_0)^3 \\
(1/6)(x - \tau_0)^3
\end{array} \right)
$$

$$
\equiv \frac{1}{6} \left( \begin{array}{c}
b^3 \\
a^3 + 6d^2b - 2b^3 \\
b^3 + 6d^2a - 2a^3 \\
a^3
\end{array} \right).
$$

**Proof.** We start with the identity in the first line of the proposition and prove that it equals the second line. To do this, we change the basis from...
$B \equiv (B_{-3}, B_{-2}, B_{-1}, B_0)$ to the new basis

$$C(x) \equiv \begin{pmatrix} C_1(x) \\ C_2(x) \\ C_3(x) \\ C_4(x) \end{pmatrix} = \begin{pmatrix} 1 \\ (\tau_1 - x) \\ (\tau_1 - x)(x - \tau_0) \\ (\tau_1 - x)(x - \tau_0)(x - \tau) \end{pmatrix}.$$}

Some calculation shows that $B = TC$ where

$$T = \frac{1}{12} \begin{pmatrix} 0 & 2d^2 & -3d & 2 \\ 2d^3 & 6d^2 & 3d & -6 \\ 8d^3 & -6d^2 & 3d & 6 \\ 2d^3 & -2d^2 & -3d & -2 \end{pmatrix}$$

with $d \equiv \tau_1 - \tau_0$ as before. Therefore we can write,

$$\tilde{H}_n(x) = \alpha' B(x) = \alpha' TC(x)$$

which identifies the coefficients of the basis $C$ as $\alpha' T$ with $\alpha$ as in (5.61).

Further algebra (we used Mathematica here) yields

$$\alpha' T = \begin{pmatrix} \mathcal{Y}_n(\tau_1) \\ -(\mathcal{Y}_n(\tau_1) - \mathcal{Y}_n(\tau_0))/d \\ -(\mathcal{Y}'_n(\tau_1) - \mathcal{Y}'_n(\tau_0))/(2d) \\ -\{\{\mathcal{Y}_n'(\tau_1) - \mathcal{Y}_n'(\tau_0)\}d + 2(\mathcal{Y}_n(\tau_1) + \mathcal{Y}_n(\tau_0))\}/d^3 \end{pmatrix},$$

and this implies that the claimed identity holds. \hfill \blacksquare

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**References.**


