Statistics 581, Autumn Quarter 2007
Problem Set 1

Reading: Please review the Spring Quarter 2007 Mathematics 576 lecture notes.

Problem 1 (review of measure theory, 4 points). Give one example each of

(a) a sequence $(\mathcal{A}_k)_{k=1,2,\ldots}$ of σ-algebras on a carefully specified set Ω such that $\mathcal{A}_k \subset \mathcal{A}_{k+1}$ for $k = 1, 2, \ldots$, with all inclusions being strict;

(b) a sequence $(f_k)_{k=1,2,\ldots}$ of measurable functions on a carefully specified measure space $(\Omega, \mathcal{A}, \mu)$ that converges almost surely and is such that all f_k are integrable but $f = \lim_{k\to\infty} f_k$ is not;

(c) a sequence $(X_k)_{k=1,2,\ldots}$ of random variables on a carefully specified probability space $(\Omega, \mathcal{A}, P)$ that converges in distribution but not in probability;

(d) a sequence $(F_k)_{k=1,2,\ldots}$ of distribution functions on the real line $\mathbb{R}$ that converges pointwise to a limit function $F = \lim_{k\to\infty} F_k$ which is not a distribution function.

Problem 2 (Riemann integral versus Lebesgue integral, 4 points). You are revising Chapter 1 of your best selling Theoretical Statistics textbook. You just introduced the notion of the integral of a nonnegative, measurable function f on a general measure space $(\Omega, \mathcal{A}, \mu)$. Now you are turning to the particular case in which $\Omega = [0, 1]$, $\mathcal{A}$ is the Borel σ-algebra and μ is the Lebesgue measure. Add a paragraph or two that explains the difference between the classical *Riemann integral* and the (just defined) *Lebesgue integral*.

Note that this is a writing exercise as much as it is a mathematical exercise. Avoid the use of abbreviations and formulas, write in complete, reasonably grammatical sentences and take advantage of graphs and illustrations. There is no need to define the Riemann integral or the Lebesgue integral; the former is known to your audience from prior course work and the latter has just been defined. Emphasize the conceptual difference between the two types of integral, and give an example of a function that is Lebesgue integrable but not Riemann integrable.

Feel free to look at existing textbooks for inspiration, but do not copy. Do not write more than a single typewritten page, excluding graphs and illustrations.

Problem 3 (probability integral transform, 4 points).

(a) Let X be a real-valued random variable with distribution function F on the real line $\mathbb{R}$. If F is continuous, show that $U = F(X)$ is standard uniform.

(b) The above fact can be used for calibration checks. Suppose that a forecaster issues continuous predictive distributions $F_1, \ldots, F_n$ for future observations $X_1, \ldots, X_n$. The respective *probability integral transform* is defined as

$$U_i = F_i(X_i) \quad \text{for} \quad i = 1, \ldots, n.$$

For an ideal calibrated forecaster $X_i \sim F_i$ holds, which implies that U_i is uniform. In practice, this may or may not be true, and we can check uniformity by plotting the histogram of the empirical probability integral transform (PIT) values $U_1, \ldots, U_n$. Often, this is done informally by visual inspection, and departures from uniformity are interpreted diagnostically. Of course, formal tests can be employed as well.

Design and implement a simulation study that applies the PIT histogram to compare two or more competing forecasters. Comment on the results and the potential use of the PIT histogram as a diagnostic device.

(c) Suppose that X and F are purely discrete, so that F has jumps of size p_k at a finite or countable collection of points $x_k \in \mathbb{R}$. Without loss of generality, suppose that $x_k < x_{k+1}$ for all k. Is the probability integral transform $U = F(X)$ standard uniform? If yes, find a proof. If no, can you think of a suitably modified version of the probability integral transform that is standard uniform?

Problem 4 (spread order, 4 points). Given a distribution function F on the real line $\mathbb{R}$, let its *pseudoinverse* F^{-1} be defined by $F^{-1}(t) = \inf\{x \in \mathbb{R} : F(x) \geq t\}$ for $t \in (0, 1)$. Then, a distribution G is *at least as spread out* than a distribution F if

$$G^{-1}(v) - G^{-1}(u) \geq F^{-1}(v) - F^{-1}(u) \quad \text{for all} \quad 0 < u < v < 1,$$

that is, if any two quantiles of G are at least as far apart as the respective quantiles of F.

(a) Show that any distribution is at least as spread out than a point measure.

(b) Suppose that F^{-1} and G^{-1} are differentiable. Show that G is at least as spread out than F if and only if

$$\frac{\mathrm{d}}{\mathrm{d}u}\left[G^{-1}(u)\right] \geq \frac{\mathrm{d}}{\mathrm{d}u}\left[F^{-1}(u)\right] \quad \text{for all} \quad 0 < u < 1,$$

and rewrite the condition by evaluating the derivatives. Hence, G is at least as spread out than F if and only if F^{-1} is nowhere steeper than G^{-1}.

(c) Suppose that F and G are strictly increasing. Show that G is at least as spread out than F if and only if there exists a strictly increasing function h such that $x' \geq x$ implies $h(x') - h(x) \geq x' - x$ and the following holds:

If X has distribution F then $h(X)$ has distribution G.

In other words, G is at least as spread out than F if and only if one can get G from F by spreading all pairs of points further apart.

(d) Suppose that F is standard uniform. Let G be a distribution with density g. Find a sufficient condition for G to be at least as spread out than F.

(e) Suppose that F is the double exponential distribution with density $f(x) = \frac{1}{2}\exp(-|x|)$. Let G be a distribution with density g. Find a sufficient condition for G to be at least as spread out than F.

(f) Discuss another example of your choice.

Tilmann Gneiting, September 28, 2007. Solutions are due Friday, October 5 at the beginning of the class session.