Statistics 581/582, Winter Quarter 2008
Problem Set 12

Reading: Ferguson, Sections 16–20.

Problem 43 (consistency of optimum score estimators, 4 points). Suppose that ν is a σ-finite measure on a measurable space $(\Omega, \mathcal{A})$. Let $\mathcal{P} = \{p_\theta : \theta \in \Theta\}$ be a parametric family of probability density functions with respect to ν. Let $X_1, \ldots, X_n$ be a sample from the density p_θ where $\theta \in \Theta$ is unknown. If S is a proper scoring rule, define

$$\bar{S}_n(\theta) = \frac{1}{n} \sum_{i=1}^{n} S(p_\theta, X_i).$$

An *optimum score estimate* of θ associated with the scoring rule S is any function $\hat{\theta}_n = \hat{\theta}_n(X_1, \ldots, X_n)$ such that

$$\bar{S}_n(\hat{\theta}_n) = \sup_{\theta \in \Theta} \bar{S}_n(\theta).$$

This definition includes the maximum likelihood estimate as the special case in which the scoring rule is logarithmic.

State and prove a theorem which is similar to our key result on the consistency of maximum likelihood estimates, but applies to optimum score estimates in general.

Problem 44 (optimum score estimators, 2 points). Give an interesting example of an optimum score estimator that is distinct from the maximum likelihood estimator.

Problem 45 (Fisher information for a location parameter, 6 points). Consider a univariate location family $f_\theta(x) = p(x - \theta)$, $\theta \in \mathbb{R}$ of densities with respect to the Lebesgue measure on the real line. Suppose that

(i) the density p is continuously differentiable,

(ii) the variance $\sigma^2 = \int_{-\infty}^{\infty} x^2 p(x) \, \mathrm{d}x$ is finite, and

(iii) $|x| \, p(x) \to 0$ as $|x| \to \infty$.

Let $\mathcal{F}_p(\theta)$ denote the Fisher information.

(a) Show that $\mathcal{F}_p(\theta) = \mathcal{F}_p$ does not depend on the parameter $\theta \in \mathbb{R}$.

(b) Show that within the class of the densities p with given variance σ^2 and satisfying conditions (i) and (iii) above,

$$\mathcal{F}_p^{-1} \leq \sigma^2$$

with equality if and only if p is a normal density.

(c) Shows that for any two numbers $0 < \mathcal{F}_p^{-1} \leq \sigma^2$ there exists a density p with Fisher information $\mathcal{F}_p$ and variance σ^2.

(d) Give an interpretation of the results in parts (b) and (c) in the context of parameter estimation.

Hint: In part (b), use partial integration to show that $\int_{-\infty}^{\infty} x p'(x) \, \mathrm{d}x = -1$ and apply the Cauchy-Schwarz inequality. In part (c), you might consider the exponential power density with scale parameter $\lambda > 0$ and shape parameter $\alpha > 1$. Use symbolic and numerical computing as appropriate.

Problem 46 (Fisher information for the bivariate normal distribution, 4 points). Consider the bivariate normal distribution with parameter vector $\theta = (\mu_1, \mu_2, \sigma_1, \sigma_2, \rho)'$ and the usual identifications. Use symbolic computing and symmetry arguments to find the Fisher information matrix and its inverse.

Tilmann Gneiting, February 1, 2008. Solutions are due Monday, February 11 at the beginning of the class session.