Statistics 581/582, Winter Quarter 2008
Problem Set 13

Reading: Ferguson, Sections 16–20.

Problem 47 (Fisher information for a location-scale family, 4 points). Consider the *location-scale* family

$$f(x,\theta) = \frac{1}{\sigma}\, p\left(\frac{x-\mu}{\sigma}\right)$$

where $\theta = (\mu,\sigma)' \in \mathbb{R}\times(0,\infty)$ and p is a smooth and strictly positive density function with respect to the Lebesgue measure on $\mathbb{R}$.

(a) Compute the Fisher information matrix for the parameter vector θ. The components of the information matrix will depend on p and σ, but not on μ.

(b) Show that the off-diagonal elements vanish if the density p is symmetric. Give an example in which p is not symmetric and the off-diagonal elements do not vanish. Give an interpretation in terms of estimation.

Problem 48 (estimation for a gamma population, 4 points.) Consider a sample of size n from the gamma distribution with Lebesgue density

$$f(x,\alpha,\beta) = \frac{1}{\Gamma(\alpha)\beta^\alpha}\, x^{\alpha-1}\, e^{-x/\beta}\, \mathbf{1}_{(0,\infty)}(x),$$

where α and β are strictly positive parameters.

(a) Find the likelihood equation and the asymptotic distribution of the maximum likelihood estimates. Note that the solution will involve the digamma and the trigamma function.

(b) If α is known, show that $\hat\beta = \bar X_n/\alpha$ is an unbiased estimate of β that attains the Cramér-Rao lower bound. What is the Cramér-Rao bound for an unbiased estimate of β when α is unknown? Compare.

Problem 49 (estimation for a beta population, 4 points). Consider a sample of size n from the Beta$(\theta,1)$ distribution with Lebesgue density

$$f(x,\theta) = \theta x^{\theta-1}\mathbf{1}_{(0,1)}(x),$$

where $\theta\in\Theta=(0,\infty)$.

(a) Find a maximum likelihood estimate of θ and its asymptotic distribution.

(b) Show that the maximum likelihood estimate of $1/\theta$ is unbiased and attains the Cramér-Rao bound.

(c) Show that $\bar{X}_n$ is an unbiased estimate of $\theta/(\theta+1)$. Compare its variance to the Cramér-Rao bound and to the asymptotic variance of the maximum likelihood estimate.

Problem 50 (maximum likelihood estimation for a normal mixture model, 4 points). Let $X_1, \ldots, X_n$ be a collection of independent identically distributed random variables, each with probability $p \in (0, 1)$ from $\mathcal{N}(\mu_1, \sigma_1^2)$ and with probability $1 - p$ from $\mathcal{N}(\mu_2, \sigma_2^2)$.

(a) Does a maximum likelihood estimate for $\theta = (p, \mu_1, \mu_2, \sigma_1, \sigma_2)'$ exist?

(b) How would you construct an asymptotically efficient sequence of estimates? Explain in a sentence or two.

(c) When $\mu_2 = 0$ and $\sigma_1 = \sigma_2 = 1$ are known, give an explicit construction of an asymptotically efficient sequence of estimates for $\theta = (p, \mu_1)'$.

Tilmann Gneiting, February 15, 2008. Solutions are due Friday, February 22 at the beginning of the class session.