Statistics 581/582, Winter Quarter 2008
Problem Set 15

Reading: Ferguson, Sections 22–24.

Problem 53 (likelihood ratio, Wald and Rao test, 6 points.) In the situation of the section on likelihood ratio tests, consider a *simple* null hypothesis

$$H_0 \; : \; \theta = \theta_0 \in \mathbb{R}^k.$$

We will study tests of H_0 that are based on the *likelihood ratio statistic*,

$$-2 \log \lambda_n = 2 \log L_n(\hat{\theta}_n) - 2 \log L_n(\theta_0);$$

the *Wald statistic*,

$$W_n = n\,(\hat{\theta}_n - \theta_0)'\,\mathcal{F}(\theta_0)\,(\hat{\theta}_n - \theta_0);$$

and the *Rao score statistic*,

$$R_n = \frac{1}{n}\,\dot{\ell}_n(\theta_0)'\,\mathcal{F}(\theta_0)^{-1}\,\dot{\ell}_n(\theta_0);$$

respectively.

(a) Review the proof of Theorem 2 and conclude that the three statistics are asymptotically equivalent under the null hypothesis and converge to a χ_k^2 random variable in distribution. Explain to which parts and equations in the proof you refer. Give references to the lecture notes.

(b) Suppose that the usual regularity conditions hold. Show that under a fixed alternative $\theta \neq \theta_0$ these statistics satisfy

$$\begin{aligned}
\frac{1}{n}\,(-2\log\lambda_n) &\longrightarrow_P \;\; 2\,K(p_\theta, p_{\theta_0}), \\
\frac{1}{n} W_n &\longrightarrow_P \;\; (\theta - \theta_0)'\,\mathcal{F}(\theta_0)\,(\theta - \theta_0), \\
\frac{1}{n} R_n &\longrightarrow_P \;\; E_\theta \psi(X, \theta_0)'\,\mathcal{F}(\theta_0)^{-1}\,E_\theta \psi(X, \theta_0),
\end{aligned}$$

respectively, where K is the Kullback-Leibler information and ψ is the score function.

(c) Is it valid to approximate the power of the Wald and Rao score tests by the method of Remark 3(a)? Discuss in one or two sentences.

(d) Suppose that $(n_1, \ldots, n_c)'$ are the observed cell frequencies in a multinomial experiment. Consider testing the hypothesis

$$H_0 \,:\, p_j = p_j^0 \qquad \text{for} \qquad j = 1, \ldots, c.$$

Find the likelihood ratio, Wald, and Rao score test statistics. Do any of these coincide with Pearson's chi-square?

(e) Design and implement a simulation study in the context of part (d). For example, you might compare the finite sample behavior of the test(s), with the goal of making recommendations for applied work. Or you might study the asymptotic behavior discussed in parts (b) and (c). Be creative, address a problem that is important and interesting, and write a report of no more than three pages, excluding tables and figures.

Problem 54 (minimum distance estimates, 4 points). Work out the details for Example 3(b) on minimum distance estimates and related tests. In particular, identify the integers k and l, the parameter space Θ, the random vector Z_n, the vector $A(\theta)$, and the matrices $\dot{A}(\theta)$, $C(\theta)$ and $M(\theta)$. Find $\nu = \text{rk}(C(\theta))$ and check whether assumptions (A1), (A2), (A3), (M1) and (M2) are satisfied.

Problem 55 (game of morra, 4 points). Study the handout on basic game theory; then do the following problem.

(a) In the game of morra, each player reveals either one or two fingers, with the winnings being the total number of fingers shown. If the total is odd, player I wins; otherwise player II wins. Discuss strategies for the players, and assess whether the game is fair.

(b) A variant is played as follows. Each player shows one or two fingers, and predicts how many fingers the other will show. If a player's prediction realizes, the other player pays her an amount equal to the total number of fingers shown. Discuss strategies for the players, and assess whether the game is fair.

Tilmann Gneiting, February 29, 2008. Solutions are due Friday, March 7 at the beginning of the class session.