Statistics 581, Autumn Quarter 2007
Problem Set 3

Reading: Ferguson, Sections 6 and 7.

Problem 9 (complications of the fourth moment, 4 points). For a random variable X, let $\mu(X)$ denote its mean, $\mu_2(X)$ its variance, and $\mu_3(X)$ and $\mu_4(X)$ its third and fourth central moment, respectively.

(a) Let X and Y be independent random variables with finite fourth moment. Show that the additive property

$$\mu_k(X + Y) = \mu_k(X) + \mu_k(Y)$$

is true for $k = 2$ and $k = 3$, but fails to hold for $k = 4$. Express $\mu_4(X + Y)$ as a function of the moments of X and Y.

(b) Let $\bar{X}$ denote the arithmetic mean of n independent random variables, each distributed like X. Show that the formula

$$\mu_k(\bar{X}) = \frac{\mu_k(X)}{n^{k-1}}$$

is true for $k = 2$ and $k = 3$, but fails to hold for $k = 4$. Express $\mu_4(\bar{X})$ as a function of $\mu_2(X)$ and $\mu_4(X)$.

(c) Let $X_1, \ldots, X_n$ be a sample of n independent observations, each distributed like X. For $k = 2$ and $k = 3$ find a real number $c_k(n)$ such that

$$Y_k = c_k(n) \sum_{i=1}^{n} \left(X_i - \bar{X} \right)^k$$

is an unbiased estimator for $\mu_k(X)$. Show that the fourth central moment $\mu_4(X)$ does not allow for an unbiased estimator that is of the above form.

Problem 10 (modes of convergence, 4 points).

(a) Show that if (X_i) is a sequence of almost surely monotonically increasing random variables, satisfying $P(X_{i+1} \geq X_i) = 1$ for $i = 1, 2, \ldots$, then convergence in probability implies almost sure convergence.

(b) Give an example of a sequence (X_i) on a carefully defined probability space that converges in probability but not almost surely.

(c) Give an example of a sequence (X_i) on a carefully defined probability space that converges almost surely but not in quadratic mean.

Problem 11 (central limit theorem precludes convergence in probability, 4 points). Let (X_i) be a sequence of independent identically distributed random variables with mean 0 and variance 1. By the Central Limit Theorem,

$$\frac{S_n}{\sqrt{n}} = \frac{X_1 + \cdots + X_n}{\sqrt{n}}$$

converges weakly to a standard normal random variable. Show that there is no random variable Y such that $S_n/\sqrt{n} \xrightarrow{P} Y$.

Problem 12 (distribution functions in $\mathbb{R}^2$, 4 points). Let $X = (X_1, X_2)'$ be a bivariate random vector on $(\Omega, \mathcal{A}, P)$.[1] For $x = (x_1, x_2)'$ and $y = (y_1, y_2)$ in $\mathbb{R}^2$ write $x \le y$ if $x_1 \le y_1$ and $x_2 \le y_2$. The *cumulative distribution function* $F : \mathbb{R}^2 \to \mathbb{R}$ of the random vector X is then defined as $F(x) = P(X \le x)$ for $x \in \mathbb{R}^2$.

(a) Show that for all $x = (x_1, x_2)' \le y = (y_1, y_2)'$ in $\mathbb{R}^2$,

$$F(y_1, y_2) - F(x_1, y_2) - F(y_1, x_2) + F(x_1, x_2) \ge 0.$$

Draw a picture and explain.

(b) Show that F is continuous from the right in that $\lim_{y_1 \downarrow x_1,\, y_2 \downarrow x_2} F(y_1, y_2) = F(x_1, x_2)$ for all $x = (x_1, x_2) \in \mathbb{R}^2$.

(c) Show that $\lim_{x_1 \to -\infty} F(x_1, z) = \lim_{x_2 \to -\infty} F(z, x_2) = 0$ for all $z \in \mathbb{R}$, and prove that $\lim_{x_1 \to \infty,\, x_2 \to \infty} F(x_1, x_2) = 1$.

(d) Find the cumulative distribution function F of a random vector that is uniform on the unit square in $\mathbb{R}^2$. Verify properties (a), (b) and (c).

(e) How would you prove that any function $F : \mathbb{R}^2 \to \mathbb{R}$ that satisfies properties (a), (b) and (c) uniquely defines a probability measure on $(\mathbb{R}^2, \mathcal{B}^2)$? And how does this probability measure relate to the random vector X? Sketch the argument in no more than a single paragraph, drawing analogies and connections to results and examples in Mathematics 576.

Tilmann Gneiting, October 12, 2007. Solutions are due Friday, October 19 at the beginning of the class session.

[1] We assume that X_1 and X_2 take real values only.