Statistics 581, Autumn Quarter 2007
Problem Set 4

Reading: Ferguson, Sections 6 and 7.

Problem 13 (joint distribution of sample statistics, 6 points). Suppose that a random variable X has mean μ, variance σ^2 and centered third and fourth moment μ_3 and μ_4, all of which are finite. If $X_1, \ldots, X_n$ is a sample of n independent observations, each distributed like X, let

$$\bar{X} = \frac{1}{n} \sum_{i=1}^{n} X_i \qquad \text{and} \qquad S^2 = \frac{1}{n-1} \sum_{i=1}^{n} \left(X_i - \bar{X}_n\right)^2$$

denote the classical unbiased estimators for μ and σ^2.

(a) Show that

$$\operatorname{var}(\bar{X}) = \frac{\sigma^2}{n}, \quad \operatorname{var}(S^2) = \frac{\mu_4}{n} - \frac{n-3}{n(n-1)}\,\sigma^4 \quad \text{and} \quad \operatorname{cov}(\bar{X}, S^2) = \frac{\mu_3}{n}.$$

(b) Show that

$$\sqrt{n}\left(\begin{pmatrix} \bar{X} \\ S^2 \end{pmatrix} - \begin{pmatrix} \mu \\ \sigma^2 \end{pmatrix}\right) \longrightarrow_d \mathcal{N}\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} \sigma^2 & \mu_3 \\ \mu_3 & \mu_4 - \sigma^4 \end{pmatrix}\right).$$

(c) Design and implement an interesting simulation study that confirms and illustrates the asymptotic result in part (b).

Problem 14 (integration by parts, 6 points).

(a) Suppose that U and V are nondecreasing, real-valued functions on $\mathbb{R}$, where U is left-continuous and V is right-continuous. Identify U and V with the associated Lebesgue-Stieltjes measures, and let $a, b \in \mathbb{R}$ with $a < b$. Apply Fubini's Theorem to show that

$$\int_{(a,b]} U(x)\,\mathrm{d}V(x) = U(x)V(x)\Big|_a^b - \int_{[a,b)} V(x)\,\mathrm{d}U(x).$$

Make sure to state the product space and the measurable function to which you apply the Fubini-Tonelli Theorem.

(b) Suppose that X is a nonnegative, real-valued random variable with distribution function F. Let $\varphi : [0, \infty) \to [0, \infty)$ be a nondecreasing differentiable function with $\varphi(0) = 0$. Use the partial integration formula to show that

$$E\,\varphi(X) = \int_{[0,\infty)} \varphi'(x)\,(1 - F(x))\,\lambda(\mathrm{d}x) = \int_0^\infty \varphi'(x)\,(1 - F(x))\,\mathrm{d}x,$$

where the second term is a Lebesgue integral and the third term is an improper Riemann integral. Specialize to the case in which $\varphi(t) = t^r$ where $r > 0$.

(c) If the random variable X is nonnegative and integer-valued, show that the formula in part (b) can be written as

$$E\,\varphi(X) = \sum_{k=0}^{\infty} (\varphi(k+1) - \varphi(k))\,P(X > k).$$

Simplify in the special cases in which $\varphi(t) = t$ and $\varphi(t) = t^2$.

Tilmann Gneiting, October 19, 2007. Solutions are due Friday, October 26 at the beginning of the class session.