Statistics 581, Autumn Quarter 2007
Problem Set 6

Reading: Ferguson, Sections 3–7.

Problem 19 (order of convergence, 4 points). A sequence of random variables (X_n) on a probability space $(\Omega, \mathcal{A}, P)$ is said to be *bounded in probability*, and we write

$$X_n = O_P(1),$$

if

$$\lim_{M\to\infty} \limsup_{n\to\infty} P(|X_n| \geq M) = 0.$$

If $X_n \longrightarrow_P 0$ we write $X_n = o_P(1)$. If (a_n) is a sequence of strictly positive real numbers, we write $X_n = O_P(a_n)$ if $X_n/a_n = O_P(1)$, and $X_n = o_P(a_n)$ if $X_n/a_n = o_P(1)$.

(a) Show that if $X_n \longrightarrow_d X$ for some real-valued random variable X then $X_n = O_P(1)$.

(b) Show that if $X_n = O_P(a_n)$ and $Y_n = O_P(b_n)$ then $X_n + Y_n = O_P(c_n)$ where $c_n = \max\{a_n, b_n\}$.

(c) Show that if $X_n = o_P(a_n)$ and $Y_n = o_P(b_n)$ then $X_n + Y_n = o_P(c_n)$ where $c_n = \max\{a_n, b_n\}$.

(d) Show that if $X_n = O_P(a_n)$ and $Y_n = O_P(b_n)$ then $X_n Y_n = O_P(a_n b_n)$.

(e) Show that if $X_n = O_P(a_n)$ and $Y_n = o_P(b_n)$ then $X_n Y_n = o_P(a_n b_n)$.

Problem 20 (order statistics, 4 points). Suppose that $X_1, \ldots, X_n$ is a sample of n independent observations, each with cumulative distribution function F. We refer to their ordered values

$$X_{1:n} \leq \cdots \leq X_{n:n}$$

as the *order statistics*.

(a) For $k = 1, \ldots, n$ show that $X_{k:n}$ has distribution function

$$F_{k:n}(x) = \sum_{i=k}^{n} \binom{n}{i} F(x)^i (1 - F(x))^{n-i} = k \binom{n}{k} \int_0^{F(x)} t^{k-1} (1-t)^{n-k} \, \mathrm{d}t$$

for $x \in \mathbb{R}$. If F has Lebesgue density f, find the density function $f_{k:n}$ of $F_{k:n}$.

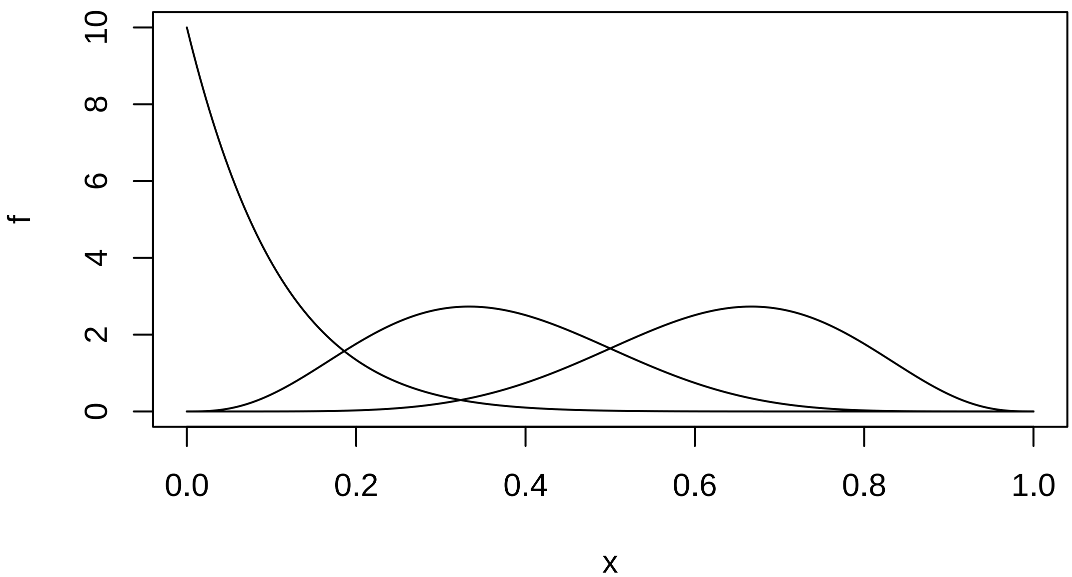

Figure 1: Densities of uniform order statistics $X_{k_1:n}, X_{k_2:n}, X_{k_3:n}$. What are the values of $k_1 < k_2 < k_3$ and n?

(b) Figure 1 shows the densities of order statistics $X_{k_1:n}, X_{k_2:n}, X_{k_3:n}$ for a sample of size n from a standard uniform population. Identify the values of $k_1 < k_2 < k_3$ and n.

(c) Suppose now that $X_1, \ldots, X_n$ is a sample from the exponential distribution with parameter λ, which has cumulative distribution function $F_\lambda(x) = 1 - \exp(-\lambda x)$ for $x \geq 0$. Then it is not unreasonable to expect the order statistic $X_{n:n}$ to be close to $b_n = F_\lambda^{-1}(1 - \frac{1}{n})$. Compute b_n explicitly and find a sequence a_n such that $X_{n:n} - b_n = O_P(a_n)$. Find a random variable Y such that $(X_{n:n} - b_n)/a_n \longrightarrow_d Y$.

Problem 21 (transformations of random variables, 4 points). Let X be a random variable on a probability space $(\Omega, \mathcal{A}, P)$. Recall that X is *discrete* if there exists a finite or countable set D such that $P(X \in D) = 1$, *continuous* if its cumulative distribution function F_X is continuous, and *absolutely continuous* with Lebesgue density f_X if

$$P(X \in (a, b]) = \int_{(a,b]} f_X \, \mathrm{d}\lambda$$

for $a, b \in \mathbb{R}$ with $a < b$, where λ is the Lebesgue measure. Suppose that $g : \mathbb{R} \to \mathbb{R}$ is a Borel measurable function and let $Y = g(X)$.

(a) Show that if X is discrete then so is Y, but not necessarily conversely.

(b) Show that if X is continuous and g is one-to-one on the range of X then Y is continuous.

(c) Suppose now that g is continuous with a strictly positive derivative g'. Show that if X is absolutely continuous then Y is also absolutely continuous with Lebesgue density

$$f_Y(y) = \frac{f_X(g^{-1}(y))}{g'(g^{-1}(y))}.$$

(d) Discuss an interesting example.

Problem 22 (scale mixtures, 4 points).

(a) Suppose that X and Y are independent random variables on a joint probability space $(\Omega, \mathcal{A}, P)$, where

$$F_Y(0) = P(Y \leq 0) = 0.$$

The random variable $Z = XY$ is called a *scale mixture* of X. Find the conditional distribution of Z given that $Y = y$ where $y > 0$. Express the cumulative distribution function F_Z of the scale mixture Z in terms of F_X and F_Y. If X has a bounded and continuous Lebesgue density f_X and $E(Y^{-1})$ is finite, show that Z is absolutely continuous with respect to Lebesgue measure, and express its density f_Z in terms of f_X and F_Y. Finally, find the characteristic function φ_Z of the scale mixture Z in terms of φ_X and F_Y.

(b) In the particular case in which X is standard normal, the random variable $Z = XY$ is called a *normal scale mixture* or a *normal/independent* variable. Specialize the above results to this case, and write down the conditional distribution of Z given that $Y = y$. Give details for at least two well-known distributions that arise as normal scale mixtures.

(c) Now suppose that X is a triangular random variable with Lebesgue density $f_X(x) = (1 - |x|)$ for $|x| \leq 1$ and $f_X(x) = 0$ otherwise. Find a necessary and sufficient condition for a random variable Z to be a scale mixture of X. Support your claim by a heuristic, possibly geometric argument and/or an analytical proof. Give interesting examples.

Tilmann Gneiting, November 2, 2007. Solutions are due Friday, November 9 at the beginning of the class session.