Rank-one latent models for cross-covariance*

Jacob A. Wegelin[†]
Thomas S. Richardson
University of Washington
Department of Statistics
Box 354322
Seattle WA 98195

David L. Ragozin
University of Washington
Department of Mathematics
Box 354350
Seattle WA 98195

August 28, 2001

Abstract

A class of Gaussian latent-variable models for cross-covariance is specified, and the set of distributions over the observed variables to which they correspond is precisely characterized. In this class the observed variables, or **indicators**, are divided into two blocks, $\mathbf{X}$ and $\mathbf{Y}$. A pair of latent variables is postulated, one for each block, $\boldsymbol{\xi}$ for $\mathbf{X}$ and $\boldsymbol{\omega}$ for $\mathbf{Y}$. The indicators are linear functions of their respective latent variables plus error, and errors for the $\mathbf{X}$ block are uncorrelated with those of the $\mathbf{Y}$ block. This latent-variable model differs from the well-known exploratory factor model in that the within-block covariances of the errors are unconstrained.

Any variance-covariance matrix over the indicators with $\mathrm{rank}(\boldsymbol{\Sigma}_{XY}) = 1$ can be fit exactly by the latent-variable model. Although the model is underidentified, the linear coefficient vectors $\mathbf{a}$, linking $\boldsymbol{\xi}$ to $\mathbf{X}$, and $\mathbf{b}$, linking $\boldsymbol{\omega}$ to $\mathbf{Y}$, are identified up to sign and scale. $\mathbf{Cor}(\boldsymbol{\xi}, \boldsymbol{\omega}) = 1$ is always feasible, and $|\mathbf{Cor}(\boldsymbol{\xi}, \boldsymbol{\omega})|$ is bounded below. When $|\mathbf{Cor}(\boldsymbol{\xi}, \boldsymbol{\omega})|$ attains its minimum, the scales of $\mathbf{a}$ and $\mathbf{b}$ are maximized and within-block errors are minimized. Subject to the constraint that $|\mathbf{Cor}(\boldsymbol{\xi}, \boldsymbol{\omega})|$ is at its minimum, the model is identified up to sign.

*Technical Report No. 391, University of Washington, Department of Statistics, Box 354322, Seattle WA 98195, U.S.A.

[†]Research supported in part by National Science Foundation Grant No. DMS-0071818. The author thanks Paul D. Sampson of the University of Washington Department of Statistics for patiently reading an earlier draft and providing valuable insights.

Contents

1 Model specification

Basic terms are introduced which will be used to state the result.

1.1 Rank-one constraint models

Let p be the number of $\mathbf{X}$-variables and q the number of $\mathbf{Y}$-variables. A **rank-one symmetric constraint model** (equivalently, a rank-one reduced-rank-regression model) is the set of $(p+q) \times (p+q)$ positive semidefinite matrices satisfying a rank constraint on the cross-covariance matrix:

$$\left.\begin{array}{l} \boldsymbol{\Sigma} \quad = \quad \begin{bmatrix} \boldsymbol{\Sigma}_{XX} & \boldsymbol{\Sigma}_{XY} \\ \boldsymbol{\Sigma}_{YX} & \boldsymbol{\Sigma}_{YY} \end{bmatrix} , \\ \\ \text{where } \boldsymbol{\Sigma}_{XY} \text{ is } p \times q \text{ of unit rank.} \end{array}\right\} \qquad (1)$$

1.2 Paired latent correlation models

A **rank-one symmetric paired latent correlation model** is the set of distributions over the latent variables $\boldsymbol{\xi}$ and $\boldsymbol{\omega}$, the observed variables $\mathbf{X}$ and $\mathbf{Y}$, and the errors $\boldsymbol{\epsilon}$ and $\boldsymbol{\zeta}$, specified as follows.

$$\left.\begin{array}{l} \left.\begin{array}{lcl} \mathbf{x} & = & \mathbf{a}\boldsymbol{\xi} + \boldsymbol{\epsilon} , \\ \mathbf{y} & = & \mathbf{b}\boldsymbol{\omega} + \boldsymbol{\zeta} , \end{array}\right\} \text{where} \\ \mathbf{Var}\begin{bmatrix} \boldsymbol{\xi} \\ \boldsymbol{\omega} \end{bmatrix} = \begin{bmatrix} 1 & \rho \\ \rho & 1 \end{bmatrix} , \\ \mathbf{Var}(\boldsymbol{\epsilon}) = \boldsymbol{\Sigma}_{\epsilon\epsilon} , \\ \mathbf{Var}(\boldsymbol{\zeta}) = \boldsymbol{\Sigma}_{\zeta\zeta} , \\ \boldsymbol{\epsilon} \perp\!\!\!\perp \begin{bmatrix} \boldsymbol{\xi} \\ \boldsymbol{\omega} \end{bmatrix} , \quad \boldsymbol{\epsilon} \perp\!\!\!\perp \boldsymbol{\zeta} , \quad \begin{bmatrix} \boldsymbol{\xi} \\ \boldsymbol{\omega} \end{bmatrix} \perp\!\!\!\perp \boldsymbol{\zeta} , \\ \mathbf{a} \in I\!\!R^p , \quad \mathbf{b} \in I\!\!R^q . \end{array}\right\} \qquad (2)$$

Thus the parameters are ρ, $\mathbf{a}$, $\mathbf{b}$, $\boldsymbol{\Sigma}_{\epsilon\epsilon}$, and $\boldsymbol{\Sigma}_{\zeta\zeta}$, subject to the constraints that $|\rho| \leq 1$ and that $\boldsymbol{\Sigma}_{\epsilon\epsilon}$ and $\boldsymbol{\Sigma}_{\zeta\zeta}$ must be positive semidefinite. The observed variables $\mathbf{X}$ and $\mathbf{Y}$ are called **indicators**, and the vectors $\mathbf{a}$ and $\mathbf{b}$ are called **saliences** or **loadings**.

A path diagram for a paired latent model may be seen in Figure 1 on page 4.

Lack of identifiability. The paired latent correlation model is underidentified. That is, in general there may be an infinite number of values of the

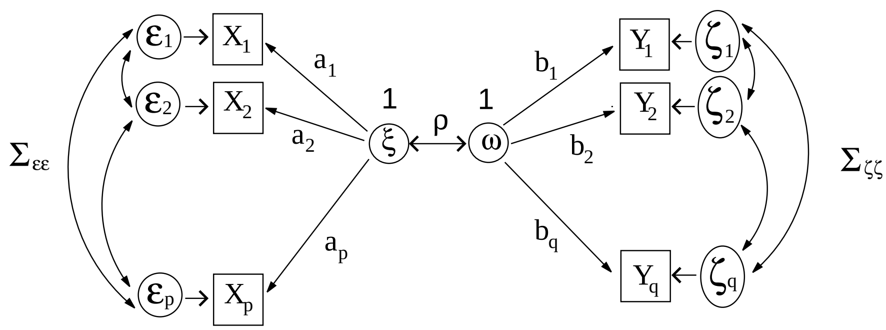

Figure 1: Path diagram of a paired latent correlation model. Paired latent correlation models are defined on page 3, Section 1.2.

full parameter set $\{\rho, \mathbf{a}, \mathbf{b}, \mathbf{\Sigma}_{\epsilon\epsilon}, \mathbf{\Sigma}_{\zeta\zeta}\}$ which induce the same distribution in the constraint model. This fact will be demonstrated in the proof of Theorem 2.1. We shall precisely characterize the degree of non-identifiability, however, and suggest a natural convention which makes the model identifiable.

1.3 Single latent models

A **rank-one symmetric single latent model** is equivalent to a paired latent model where $\boldsymbol{\xi} \equiv \boldsymbol{\omega}$. It is the set of distributions over the latent variable $\boldsymbol{\eta}$, the errors $\boldsymbol{\epsilon}$ and $\boldsymbol{\zeta}$, and the observed variables $\mathbf{X}$ and $\mathbf{Y}$, specified as follows.

$$\left.\begin{array}{rcl} \mathbf{x} & = & \mathbf{a}\boldsymbol{\eta} + \boldsymbol{\epsilon} \ , \\ \\ \mathbf{y} & = & \mathbf{b}\boldsymbol{\eta} + \boldsymbol{\zeta} \ , \end{array}\right\} \text{where}$$

$$\begin{array}{rcll} \mathbf{Var}\,(\boldsymbol{\eta}) & = & 1 \ , & \\ \mathbf{Var}(\boldsymbol{\epsilon}) & = & \mathbf{\Sigma}_{\epsilon\epsilon}, & p \times p, \\ \mathbf{Var}(\boldsymbol{\zeta}) & = & \mathbf{\Sigma}_{\zeta\zeta}, & q \times q, \end{array}$$

$$\boldsymbol{\epsilon} \perp\!\!\!\perp \boldsymbol{\eta} \ , \quad \boldsymbol{\epsilon} \perp\!\!\!\perp \boldsymbol{\zeta} \ , \quad \boldsymbol{\eta} \perp\!\!\!\perp \boldsymbol{\zeta} \ ,$$

$$\mathbf{a} \in I\!\!R^p \ , \quad \mathbf{b} \in I\!\!R^q \ .$$

Thus the parameters of a symmetric single latent model are $\mathbf{\Sigma}_{\epsilon\epsilon}$, $\mathbf{\Sigma}_{\zeta\zeta}$, $\mathbf{a}$ and $\mathbf{b}$, where $\mathbf{\Sigma}_{\epsilon\epsilon}$ and $\mathbf{\Sigma}_{\zeta\zeta}$ must be positive semidefinite. A path diagram is seen in Figure 2 on page 5.

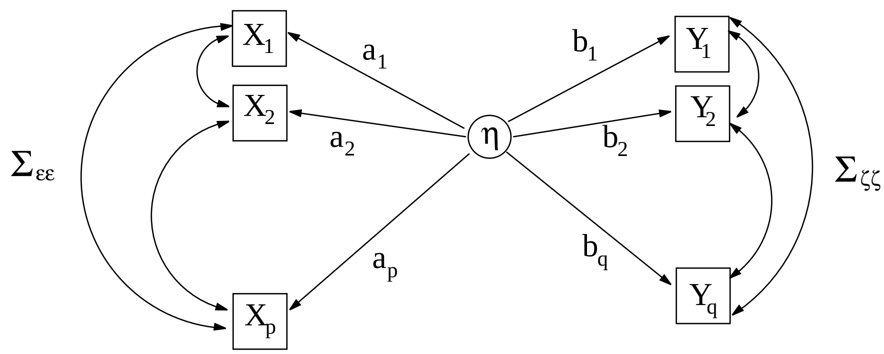

Figure 2: Path diagram of a rank-one single latent model, discussed on page 4, Section 1.3.

2 Maps between spaces of models

Every set of parameter values in a rank-one paired latent correlation model induces a distribution in the rank-one constraint model as follows:

$$\left.\begin{array}{rcl} \mathbf{\Sigma}_{XX} & = & \mathbf{a}\mathbf{a}^T + \mathbf{\Sigma}_{\epsilon\epsilon} \ , \\ \mathbf{\Sigma}_{YY} & = & \mathbf{b}\mathbf{b}^T + \mathbf{\Sigma}_{\zeta\zeta} \ , \\ \mathbf{\Sigma}_{XY} & = & \mathbf{a}\mathbf{b}^T \rho \ . \end{array}\right\} \tag{3}$$

The equations (3) define a map from the space of symmetric rank-one paired latent correlation model parameterizations into the space of rank-one constraint model distributions. The existence of such a map immediately raises the question whether every distribution in the rank-one constraint model can be obtained by a set of parameter values in a paired latent correlation model—i.e., is the map onto. If such a set of parameters may be found for a given distribution in the constraint model, we shall say that this set **parameterizes** or **is a paired latent parameterization of** the distribution over the constraint model.

The answer to the question in the previous paragraph is yes. Every rank-one constraint model can be parameterized by a symmetric paired latent correlation model. We show this by first proving a stronger result, i.e., that any rank-one constraint model can be parameterized by a symmetric single latent model. The result regarding paired latent correlation models is then obtained as a corollary.

2.1 A theorem regarding single latent models

We now state and prove the main result.

Theorem 2.1 *For each distribution within the rank-one constraint model there is a non-void class of parameter values in the symmetric single latent model which induce this distribution.*

Proof. We use two lemmas, stated and proved in Section 5.1. Decompose $\mathbf{\Sigma}$ as follows:

$$\mathbf{Q} = \begin{bmatrix} \mathbf{a}\mathbf{a}^T & \mathbf{a}\mathbf{b}^T \\ \mathbf{b}\mathbf{a}^T & \mathbf{b}\mathbf{b}^T \end{bmatrix} , \tag{4}$$

$$\tag{5}$$

$$\mathbf{E} = \begin{bmatrix} \mathbf{\Sigma}_{\epsilon\epsilon} & \mathbf{0} \\ \mathbf{0} & \mathbf{\Sigma}_{\zeta\zeta} \end{bmatrix} ,$$

so that

$$\mathbf{\Sigma} = \mathbf{Q} + \mathbf{E} .$$

Given a covariance $\mathbf{\Sigma}$ as in (1), that is, a distribution under the rank-one constraint model, we seek $\mathbf{a}$, $\mathbf{b}$, $\mathbf{\Sigma}_{\epsilon\epsilon}$, and $\mathbf{\Sigma}_{\zeta\zeta}$ such that

$$\left.\begin{array}{rcll} \mathbf{\Sigma}_{XX} & = & \mathbf{a}\mathbf{a}^T + \mathbf{\Sigma}_{\epsilon\epsilon} , & \mathbf{\Sigma}_{\epsilon\epsilon} \text{ positive semidefinite,} \\ \mathbf{\Sigma}_{YY} & = & \mathbf{b}\mathbf{b}^T + \mathbf{\Sigma}_{\zeta\zeta} , & \mathbf{\Sigma}_{\zeta\zeta} \text{ positive semidefinite,} \end{array}\right\} , \tag{6}$$

$$\text{and } \mathbf{\Sigma}_{XY} = \mathbf{a}\mathbf{b}^T . \tag{7}$$

Since $\mathbf{\Sigma}_{XY}$ has rank one, by the singular value decomposition we can always find $\mathbf{a}$ and $\mathbf{b}$ satisfying (7). The two vectors are only determined up to sign and scale, however, since for any $\delta \neq 0$,

$$\mathbf{\Sigma}_{XY} = \mathbf{a}\mathbf{b}^T \Rightarrow \mathbf{\Sigma}_{XY} = (\delta\mathbf{a}) \left(\frac{\mathbf{b}^T}{\delta} \right) .$$

The scale and sign of $\mathbf{a}$ constitute the only degree of freedom, or lack of identifiability, in the map from the constraint model to the single latent model. This is because the direction of $\mathbf{a}$ is determined by (7). Once the sign and scale of $\mathbf{a}$ are determined, then $\mathbf{b}$ is determined by (7), and $\mathbf{\Sigma}_{\epsilon\epsilon}$ and $\mathbf{\Sigma}_{\zeta\zeta}$ are determined by (6).

Let us express the single degree of freedom in this model formally. Define $\mathbf{u}$ and $\mathbf{v}$ according to the convention of the singular value decomposition. That is, let

$$\mathbf{\Sigma}_{XY} = \mathbf{u}\mathbf{v}^T d , \quad ||\mathbf{u}|| = ||\mathbf{v}|| = 1 , \tag{8}$$

where $||\cdot||$ represents the Euclidean norm. Furthermore let us assume that a sign convention has been adopted, so that the lack of identifiability consists only in the scale of $\mathbf{a}$. For $0 < \alpha$, let

$$\mathbf{a}(\alpha) \equiv \alpha\mathbf{u} , \quad \mathbf{b}(\alpha) \equiv \frac{\mathbf{v}d}{\alpha} . \tag{9}$$

For future reference we note that

$$||\mathbf{a}(\alpha)|| = \alpha , \quad ||\mathbf{b}(\alpha)|| = \frac{d}{\alpha} . \tag{10}$$

Thus $\mathbf{a}(\alpha)$ and $\mathbf{b}(\alpha)$ satisfy $\mathbf{\Sigma}_{XY} = \mathbf{a}(\alpha)\,[\mathbf{b}(\alpha)]^T$. To show that a latent parameterization exists it suffices to show that, if $\mathbf{\Sigma}$ is positive semidefinite, a value of α can always be found such that the values determined by

$$\left.\begin{array}{rcccl} \mathbf{\Sigma}_{\epsilon\epsilon}(\alpha) & \equiv & \mathbf{\Sigma}_{XX} - \mathbf{a}(\alpha)\,[\mathbf{a}(\alpha)]^T & = & \mathbf{\Sigma}_{XX} - \alpha^2\,\mathbf{u}\,[\mathbf{u}]^T \\ \mathbf{\Sigma}_{\zeta\zeta}(\alpha) & \equiv & \mathbf{\Sigma}_{YY} - \mathbf{b}(\alpha)\,[\mathbf{b}(\alpha)]^T & = & \mathbf{\Sigma}_{YY} - \dfrac{\mathbf{v}\mathbf{v}^T d^2}{\alpha^2} \end{array}\right\} \tag{11}$$

are positive semidefinite. Define $f : (0,\infty) \mapsto I\!\!R$ and $g : (0,\infty) \mapsto I\!\!R$ by

$$\begin{array}{rcl} f(\alpha) & = & \min\{\text{eigenvalues of } \mathbf{\Sigma}_{\epsilon\epsilon}(\alpha)\}\ , \\ g(\alpha) & = & \min\{\text{eigenvalues of } \mathbf{\Sigma}_{\zeta\zeta}(\alpha)\}\ . \end{array} \tag{12}$$

It may be shown that these functions are continuous (Theorem 6.3.2, page 365 of Horn and Johnson [7]). By Parts 1 and 3 of Lemma 5.1:

- f is monotone nonincreasing and goes to $-\infty$ as $\alpha \to \infty$;
- g is monotone nondecreasing and goes to $-\infty$ as $\alpha \downarrow 0$.

Let

$$\begin{array}{rcl} \mathcal{F} & = & \{\alpha : f(\alpha) < 0\}\ , \quad \text{and} \\ \mathcal{G} & = & \{\alpha : g(\alpha) < 0\}\ . \end{array}$$

By the continuity of f and g these sets are open, but by monotonicity they are in fact intervals:

$$\begin{array}{rcl} \mathcal{F} & = & (\alpha_1, \infty) \quad \text{and} \\ \mathcal{G} & = & (0, \alpha_2)\ . \end{array}$$

The closed set $I\!\!R \setminus (\mathcal{F} \cup \mathcal{G})$ is the set of feasible α values. By Lemma 5.4, this set is nonvoid; that is, we must have $\alpha_2 \leq \alpha_1$. Since this is the case, let us call them respectively $\alpha_{\min}$ and $\alpha_{\max}$. The feasible set of values for α is

$$[\alpha_{\min}, \alpha_{\max}]\ , \tag{13}$$

and we note for future reference:

$$\begin{array}{rcl} \alpha_{\min} & = & \min\left\{\alpha : \mathbf{\Sigma}_{YY} - \dfrac{\mathbf{v}\mathbf{v}^T d^2}{\alpha^2} \text{ is positive semidefinite}\right\}\ , \\ \alpha_{\max} & = & \max\left\{\alpha : \mathbf{\Sigma}_{XX} - \alpha^2 \mathbf{u}\mathbf{u}^T \text{ is positive semidefinite}\right\}\ . \end{array} \tag{14}$$

These follow from the definitions at (11) and (12).

The fact that $I\!\!R \setminus (\mathcal{F} \cup \mathcal{G})$ is nonvoid means the following: In equations (6) and (7) on page 6 there is at least one scale of the salience vector $\mathbf{a}$ such that both $\mathbf{\Sigma}_{\epsilon\epsilon}$ and $\mathbf{\Sigma}_{\zeta\zeta}$ are positive semidefinite. Thus there is a single-latent parameterization of any rank-one constraint model. □

Examples of constraint models and their parameterizations by the single latent model are presented in Section 3, starting on page 11.

Corollary 2.2 *Each constraint model can be parameterized by at least one paired latent model.*

Proof. Let $\boldsymbol{\eta}$ be the latent variable of the single latent model. Let $\boldsymbol{\xi}$ and $\boldsymbol{\omega}$ be the latent variables in the paired latent model, and let $\boldsymbol{\xi} \equiv \boldsymbol{\omega} \equiv \boldsymbol{\eta}$. □

2.2 Practical considerations

The proof of Theorem 2.1 suggests that the task of finding a single-latent parameterization of a covariance matrix (1) in the rank-one constraint model might be broken into the following two steps: First find a decomposition

$$\boldsymbol{\Sigma}_{XY} = \mathbf{a}\mathbf{b}^T \; ; \tag{15}$$

then estimate

$$\begin{aligned} \alpha_{\max} &\equiv \max\left\{\alpha : \boldsymbol{\Sigma}_{XX} - \alpha^2 \mathbf{a}\mathbf{a}^T \text{ is positive semidefinite}\right\} \\ \alpha_{\min} &\equiv \min\left\{\alpha : \boldsymbol{\Sigma}_{YY} - \mathbf{b}\mathbf{b}^T/\alpha^2 \text{ is positive semidefinite}\right\} \; . \end{aligned} \tag{16}$$

The decomposition of $\boldsymbol{\Sigma}_{XY}$. If $\boldsymbol{\Sigma}$ is known, the decomposition (15) is exact, and can be found directly. For instance, set $\mathbf{a}$ equal to the first nonzero column of $\boldsymbol{\Sigma}_{XY}$, and determine $\mathbf{b}$ by

$$\begin{aligned} \mathbf{b}_j &= \frac{(\boldsymbol{\Sigma}_{XY})_{ij}}{\mathbf{a}_i} \text{ , where} \\ i &= \min\{k : \mathbf{a}_k \neq 0\} \; . \end{aligned}$$

When $\boldsymbol{\Sigma}$ is estimated by the sample covariance matrix $\mathbf{S}$, in most cases we will have $\operatorname{rank}(\mathbf{S}_{XY}) > 1$ and (15) will be an approximation. Then standard singular value decomposition software could be used. For instance, let $\mathbf{u}$ and $\mathbf{v}$ be the first pair of singular vectors, and define $\mathbf{a}$ and $\mathbf{b}$ as in (9) on page 6.

Estimating $\alpha_{\min}$ and $\alpha_{\max}$. From (16) we have

$$\begin{aligned} \alpha_{\max} &= \max\left\{\alpha : \text{least eigenvalue of } \boldsymbol{\Sigma}_{XX} - \alpha^2 \mathbf{a}\mathbf{a}^T \text{ is nonnegative}\right\} \\ &= \max\left\{\alpha : \text{least eigenvalue of } \boldsymbol{\Sigma}_{XX} - \alpha^2 \mathbf{a}\mathbf{a}^T \text{ is zero}\right\} && (17) \\ &= \max\left\{\alpha : \left|\boldsymbol{\Sigma}_{XX} - \alpha^2 \mathbf{a}\mathbf{a}^T\right| = 0\right\} \; , && (18) \end{aligned}$$

where the equality at (17) follows by the continuity of the least eigenvalue. By Propositions 5.5 and 5.6 on page 25, to find α satisfying the expression inside the brackets at (18) we may consider the eigenvalue problem

$$\left|\boldsymbol{\Sigma}_{XX}^{-1} \mathbf{a}\mathbf{a}^T - \lambda \mathbf{I}\right| = 0 \; ,$$

provided the inverse exists. Since $\boldsymbol{\Sigma}_{XX}^{-1} \mathbf{a}\mathbf{a}^T$ is a rank-one matrix there will be a single nonzero eigenvalue, say λ. But since $\boldsymbol{\Sigma}_{XX}^{-1} \mathbf{a}\mathbf{a}^T$ is not necessarily symmetric, we have not yet confirmed that the eigenvalue will be positive or even real. Let us therefore convert the equation inside the brackets at (18) to a symmetric, positive semidefinite eigenvalue problem. Let $\mathbf{R}$ be the inverse of a symmetric positive definite square root of $\mathbf{\Sigma}_{XX}$. Since $|\mathbf{R}| > 0$, we may left-multiply and right-multiply both sides of the equation by $|\mathbf{R}|$ without changing the problem, to obtain

$$\left|\mathbf{I} - \alpha^2 \mathbf{R}\mathbf{a}\mathbf{a}^T\mathbf{R}\right| \quad = \quad 0 \; . \tag{19}$$

By Proposition 5.5, or by multiplying both sides of (19) by $\left(-\frac{1}{\alpha^2}\right)^n$, we obtain the symmetric positive semidefinite eigenvalue problem

$$\left|\mathbf{R}\mathbf{a}\mathbf{a}^T\mathbf{R} - \frac{1}{\alpha^2}\mathbf{I}\right| \quad = \quad 0 \; .$$

An example of an eigenvalue approach to finding $\alpha_{\min}$ and $\alpha_{\max}$ is presented in Section 3.

The equation inside the brackets at (18) expresses a **generalized eigenvalue problem**. The problem of computing generalized eigenvalues has been studied extensively. Computational methods exist which are preferable to the procedures presented above. A brief discussion of the generalized eigenvalue problem, an introduction to computational issues, and a bibliography, may be found in Golub and van Loan [6], pages 375–390 and 461ff.

2.3 Parameterization of paired latent correlation models

The proof of Theorem 2.1 guarantees at least one paired-latent parameterization of any rank-one constraint model. In this parameterization the latent variables ($\boldsymbol{\xi}$ for the $\mathbf{X}$ block, $\boldsymbol{\omega}$ for the $\mathbf{Y}$ block) have unit variance and unit covariance, hence unit correlation. In the current section the complete set of paired latent parameterizations for a given rank-one constraint model will be characterized.

A distribution in the rank-one constraint model is mapped to an equivalence class of parameter values in the symmetric paired latent correlation model as follows. Consider the following set:

$$\left\{(\rho, \alpha): \quad |\rho| \leq 1, \quad \alpha_{\min}^2 \leq \alpha^2\rho^2, \quad \alpha \leq \alpha_{\max}\right\} \; , \tag{20}$$

where $\alpha_{\min}$ and $\alpha_{\max}$ are defined by (14) in the proof of Theorem 2.1. When $\alpha_{\min} = \alpha_{\max}$, this set is a singleton; otherwise it is a continuous closed region. Let

$$\boldsymbol{\theta} \equiv (\rho, \mathbf{a}, \mathbf{b}, \mathbf{\Sigma}_{\epsilon\epsilon}, \mathbf{\Sigma}_{\zeta\zeta})$$

denote the parameter vector for the paired latent correlation model. Each point in (20) determines a value of $\boldsymbol{\theta}$ as follows.

$$\left.\begin{array}{rcl} \mathbf{a} & = & \alpha \mathbf{u} \ , \\ \mathbf{b} & = & \dfrac{\mathbf{v}}{\alpha\rho} d \ , \\ \boldsymbol{\Sigma}_{\epsilon\epsilon} & = & \boldsymbol{\Sigma}_{XX} - \mathbf{a}\mathbf{a}^T \ , \\ \boldsymbol{\Sigma}_{\zeta\zeta} & = & \boldsymbol{\Sigma}_{YY} - \mathbf{b}\mathbf{b}^T \ . \end{array}\right\} \qquad (21)$$

When the map (21) is applied to any value outside the set (20), the resulting values do not define feasible parameters for the paired latent correlation model. This will be shown in the current section. Consequently (20) is called the **feasible set** for the rank-one paired latent correlation model. An example of a feasible set may be seen in Figure 4 on page 14.

We have already seen in Corollary (2.2) that $\mathbf{Cor}\,(\boldsymbol{\xi}, \boldsymbol{\omega}) = 1$ is always feasible. Observing that

$$\alpha^2 \leq \alpha^2_{\max} \ \text{ and } \ \alpha^2_{\min} \leq \rho^2\alpha^2 \quad \Rightarrow \quad |\rho| \geq \frac{\alpha_{\min}}{\alpha_{\max}}$$

we see that the constraints at (20) entail a lower bound on the correlation, and we define

$$\rho_{\min} = \frac{\alpha_{\min}}{\alpha_{\max}} \ .$$

To justify the term "feasible set" for (20), it suffices to show

1. that the parameter values defined above recover $\boldsymbol{\Sigma}$ (that this parameterization maps into the constraint model with which we started), and

2. that the constraints at (20) are necessary and sufficient for the parameters to be feasible—that is, for $\boldsymbol{\Sigma}_{\epsilon\epsilon}$, $\boldsymbol{\Sigma}_{\zeta\zeta}$, and $\begin{bmatrix} 1 & \rho \\ \rho & 1 \end{bmatrix}$ to be positive semidefinite.

Condition (1), the fact that the parameterization recovers the constraint model, follows from (21) and from the fact that

$$\begin{array}{rcll} \mathbf{Cov}\,(\mathbf{x}, \mathbf{y}) & = & \mathbf{a}\mathbf{b}^T\rho & \text{by (2)} \\ & = & \mathbf{u}\mathbf{v}^T d & \text{by (21)} \\ & = & \boldsymbol{\Sigma}_{XY} & \text{by (8).} \end{array}$$

As for Condition (2), the feasibility of the set, the constraint $|\rho| \leq 1$ is necessary and sufficient for $\begin{bmatrix} 1 & \rho \\ \rho & 1 \end{bmatrix}$ to be positive semidefinite. Since

$$\boldsymbol{\Sigma}_{\epsilon\epsilon} \ = \ \boldsymbol{\Sigma}_{XX} - \mathbf{a}\mathbf{a}^T \ = \ \boldsymbol{\Sigma}_{XX} - \alpha^2\mathbf{u}\mathbf{u}^T \ ,$$

by (14) on page 7 a necessary and sufficient condition for $\mathbf{\Sigma}_{\epsilon\epsilon}$ to be positive semidefinite is $\alpha \le \alpha_{\max}$. By the definitions of $\mathbf{\Sigma}_{\zeta\zeta}$ and $\mathbf{b}$ at (21),

$$\mathbf{\Sigma}_{\zeta\zeta} \quad = \quad \mathbf{\Sigma}_{YY} - \mathbf{b}\mathbf{b}^T \quad = \quad \mathbf{\Sigma}_{YY} - \frac{\mathbf{v}\mathbf{v}^T}{\rho^2\alpha^2} d^2 \ .$$

By (14) on page 7, the greatest real number t such that $\mathbf{\Sigma}_{YY} - t\mathbf{v}\mathbf{v}^T$ is positive semidefinite is $\dfrac{d^2}{\alpha^2_{\min}}$. Thus

$$\mathbf{\Sigma}_{\zeta\zeta} \text{ is positive semidefinite} \quad \Leftrightarrow \quad \frac{d^2}{\rho^2\alpha^2} \le \frac{d^2}{\alpha^2_{\min}} \quad \Leftrightarrow \quad \alpha^2_{\min} \le \rho^2\alpha^2 \ .$$

The parameterization has been shown to be correct.

3 Examples

Parameterization of a 5×5 positive definite matrix in the rank-one constraint model. Consider the following symmetric positive definite matrix.

$$\mathbf{\Sigma} \quad = \quad \begin{bmatrix} 7 & 0 & 0 & 1 & 0.5 \\ 0 & 7 & 0 & 2 & 1 \\ 0 & 0 & 7 & 3 & 1.5 \\ 1 & 2 & 3 & 9 & 0 \\ 0.5 & 1 & 1.5 & 0 & 5 \end{bmatrix} \tag{22}$$

Let $p = 3$, $q = 2$. We shall first parameterize (22) using a simple approach which parallels the proof of Theorem 2.1. Then we shall parameterize it again, using the generalized eigenvalue approach of Section 2.2, and verify that the approaches give identical results.

Choosing the convention that both d and the component of $\mathbf{u}$ with greatest absolute value shall be positive, we obtain

$$\mathbf{a}(\alpha) \quad = \quad \frac{\alpha}{\sqrt{14}} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \quad , \quad \mathbf{b}(\alpha) \quad = \quad \frac{\sqrt{14}}{2\alpha} \begin{bmatrix} 2 \\ 1 \end{bmatrix} \quad , \quad d \quad = \quad \sqrt{\frac{35}{2}} \ ,$$

$$\mathbf{\Sigma}_{\epsilon\epsilon}(\alpha) \quad = \quad \frac{1}{14} \begin{bmatrix} 98 - \alpha^2 & -2\alpha^2 & -3\alpha^2 \\ -2\alpha^2 & 98 - 4\alpha^2 & -6\alpha^2 \\ -3\alpha^2 & -6\alpha^2 & 98 - 9\alpha^2 \end{bmatrix} \ ,$$

$$\mathbf{\Sigma}_{\zeta\zeta}(\alpha) \quad = \quad \begin{bmatrix} 9 - \dfrac{14}{\alpha^2} & -\dfrac{7}{\alpha^2} \\ -\dfrac{7}{\alpha^2} & 5 - \dfrac{7}{2\alpha^2} \end{bmatrix} \ ,$$

$$\det \mathbf{\Sigma}_{\epsilon\epsilon}(\alpha) = 343 - 49\alpha^2 \ , \quad \det \mathbf{\Sigma}_{\zeta\zeta}(\alpha) = 45 - \frac{203}{2\alpha^2} \ ,$$

$$[\alpha_{\min} \ , \ \alpha_{\max}] = \left[\sqrt{\frac{203}{90}}, \sqrt{7} \right] \approx [1.50, 2.65] \ .$$

The curves of least eigenvalues are plotted in Figure 3 on page 13. The minimum feasible correlation is $\rho_{\min} \equiv \dfrac{\alpha_{\min}}{\alpha_{\max}} = \dfrac{1}{30}\sqrt{290} \approx 0.57$. The feasible set for the paired latent correlation model is displayed in Figure 4 on page 14.

In Section 2.2 it was pointed out that the problem of finding $\alpha_{\min}$ and $\alpha_{\max}$ is a generalized eigenvalue problem. We shall now follow the derivation in Section 2.2 and verify that its solution is identical to that derived above. Following (15) on page 8, let us take $\mathbf{a}(1)$ and $\mathbf{b}(1)$ as our decomposition of the cross-covariance. To find $\alpha_{\max}$ we then solve $\left| \mathbf{\Sigma}_{XX}^{-1}\mathbf{a}(1)\mathbf{a}(1)^T - \dfrac{1}{\alpha^2}\mathbf{I} \right| = 0$, where

$$\mathbf{\Sigma}_{XX}^{-1}\mathbf{a}(1)\mathbf{a}(1)^T = \frac{1}{98}\begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 9 \end{bmatrix} .$$

The sole nonzero eigenvalue is $\dfrac{1}{7}$; thus $\alpha_{\max}^2 = 7$, as expected. Similarly we note that $\alpha_{\min}$ solves $\left| \mathbf{\Sigma}_{YY}^{-1}\mathbf{b}(1)\mathbf{b}(1)^T - \alpha^2\mathbf{I} \right| = 0$, where

$$\mathbf{\Sigma}_{YY}^{-1}\mathbf{b}(1)\mathbf{b}(1)^T = \begin{bmatrix} \frac{14}{9} & \frac{7}{9} \\ \frac{7}{5} & \frac{7}{10} \end{bmatrix} .$$

The single positive eigenvalue is $\dfrac{203}{90}$, our solution for $\alpha_{\min}^2$, as expected.

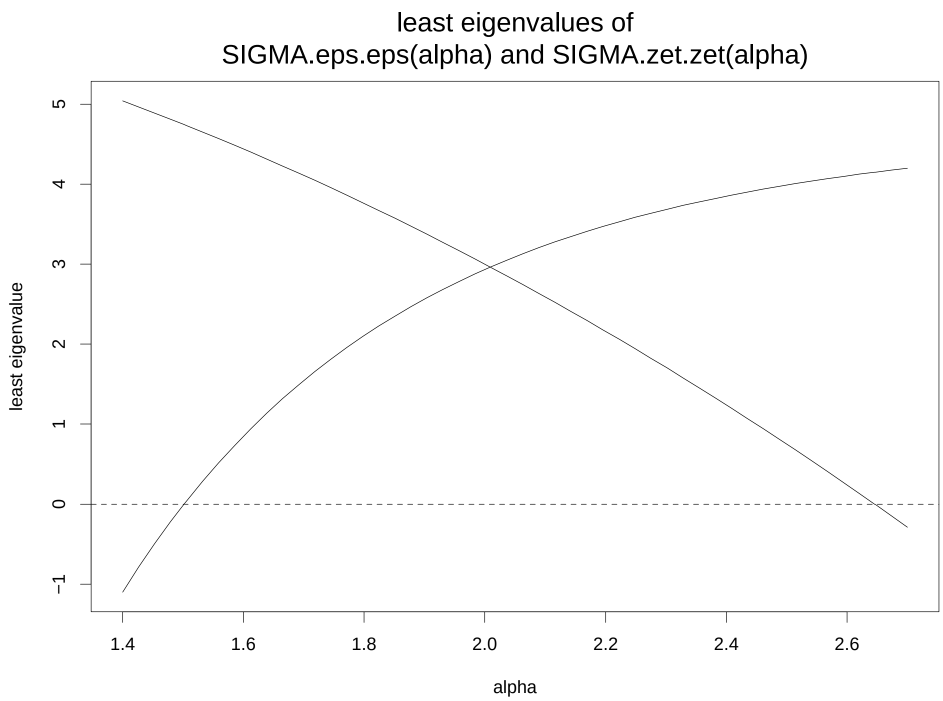

Figure 3: The least eigenvalues of $\boldsymbol{\Sigma}_{\epsilon\epsilon}(\alpha)$ (the decreasing function) and $\boldsymbol{\Sigma}_{\zeta\zeta}(\alpha)$ in the single latent parameterization of the matrix at line (22), page 11. The feasible values for α lie in the closed interval $\left[\sqrt{\frac{203}{90}}, \sqrt{7}\right] \approx [1.50, 2.65]$. These are the values of α for which the least eigenvalues of both $\boldsymbol{\Sigma}_{\epsilon\epsilon}(\alpha)$ and $\boldsymbol{\Sigma}_{\zeta\zeta}(\alpha)$ are nonnegative.

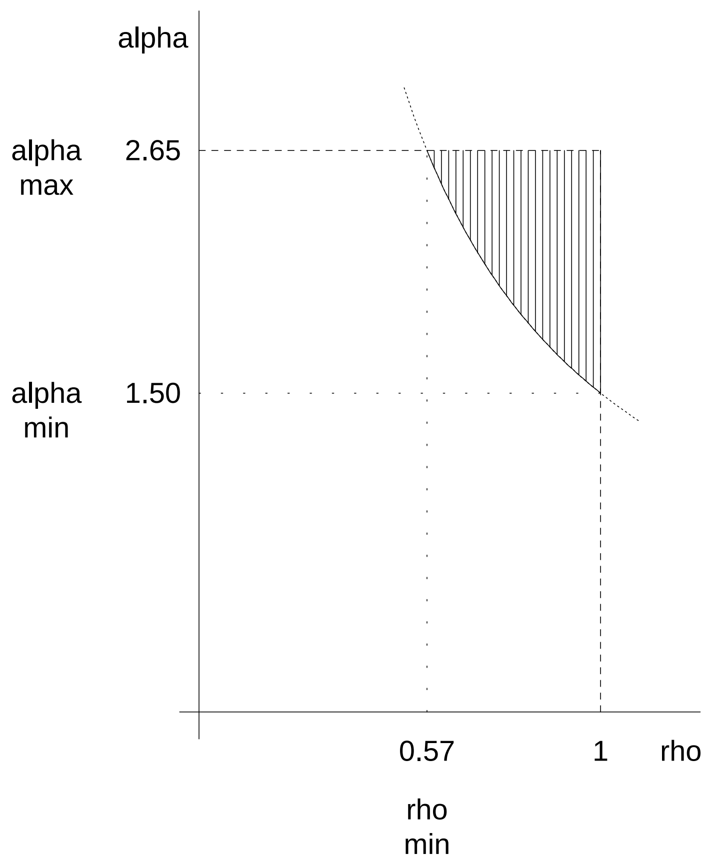

Figure 4: Feasible ρ and α for the paired-latent correlation parameterization of the rank-constraint distribution specified by (22). Feasible values are in the shaded region. The right boundary of the feasible set corresponds to the single latent model. The (curved) left boundary is the line $\rho\alpha = \alpha_{\min}$. The minimum feasible correlation is $\rho_{\min} \equiv \dfrac{\alpha_{\min}}{\alpha_{\max}}$.

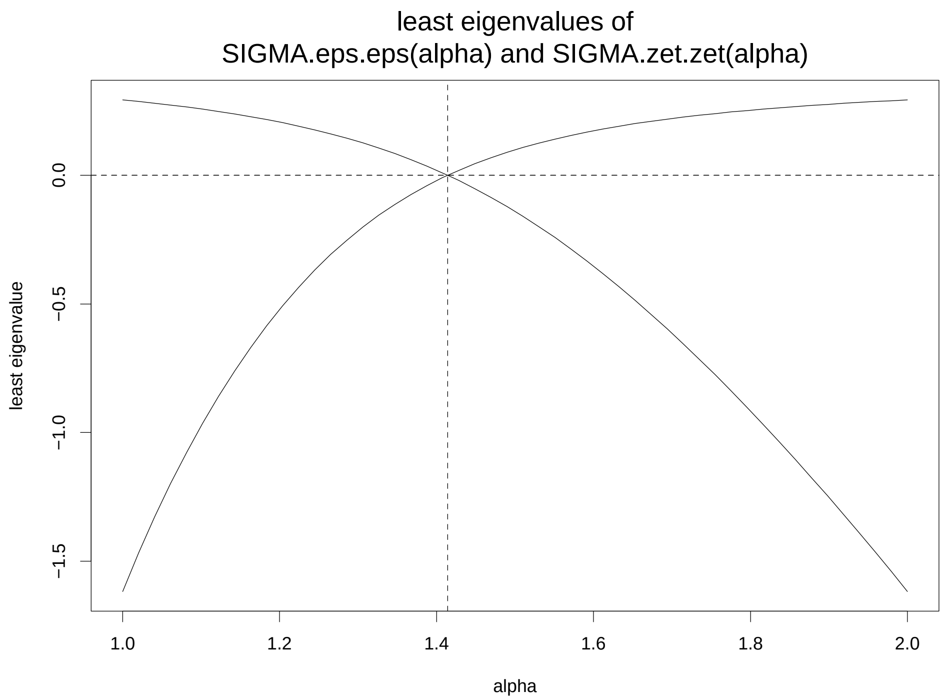

Figure 5: The least eigenvalues of $\mathbf{\Sigma}_{\epsilon\epsilon}(\alpha)$ and $\mathbf{\Sigma}_{\zeta\zeta}(\alpha)$ for the singular matrix at line (23), page 15, where there are two **X**-variables and two **Y**-variables. The decreasing function is the least eigenvalue of $\mathbf{\Sigma}_{\epsilon\epsilon}(\alpha)$. Since $\sqrt{2}$ is the only point where both curves equal or exceed zero, this is the only feasible value for α.

A singular matrix in the rank-one constraint model for which the feasible set consists of a single point. The following matrix is singular. For purposes of comparison with the matrix at (24) on page 16 we note that its eigenvalues are $\{4.56, 1, 0.44, 0\}$.

$$\mathbf{\Sigma} \;=\; \begin{bmatrix} 2 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 2 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix} \tag{23}$$

Let $p = q = 2$. Then the only value of α which parameterizes the single latent model is $\sqrt{2}$. Consequently, in the paired latent parameterization the only feasible correlation is unity. The least eigenvalues of $\mathbf{\Sigma}_{\epsilon\epsilon}(\alpha)$ and $\mathbf{\Sigma}_{\zeta\zeta}(\alpha)$ are plotted in Figure 5, page 15.

A singular matrix in the rank-one constraint model for which the feasible set is infinite. The example at (23) notwithstanding, a matrix in the rank-one constraint model may be singular and still admit an infinite number of paired-latent parameterizations. The following degenerate case illustrates this. The matrix at (24) has eigenvalues 4.56, 1, 0.44, and 0, as does the matrix at (23).

$$\mathbf{\Sigma} \quad = \quad \begin{bmatrix} 2 & 1 & 1 & 1 \\ 1 & 2 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix} \tag{24}$$

Again let $p = q = 2$. The matrix (24) represents a degenerate distribution, since the two $\mathbf{Y}$ variables are perfectly correlated. The fact that there are infinitely many feasible parameterizations follows from the fact that $\mathbf{v}\mathbf{v}^T$ is proportional to $\mathbf{\Sigma}_{YY}$. The feasible set is $\left[\sqrt{2}, \sqrt{3}\right]$. The value $\alpha = \sqrt{2}$ entails zero error for the $\mathbf{Y}$-block, so that each $\mathbf{Y}$ variable measures the latent $\boldsymbol{\eta}$ exactly. All feasible values of α, however, entail a singular error covariance for the $\mathbf{Y}$-block. For all values of α, whether feasible or not, $\mathbf{\Sigma}_{\zeta\zeta}(\alpha) = \begin{bmatrix} \delta & \delta \\ \delta & \delta \end{bmatrix}$ for some $\delta \in I\!R$. When $\sqrt{2} \leq \alpha$, so that $\delta \geq 0$, the least eigenvalue is 0. For $\alpha < \sqrt{2}$, hence $\delta < 0$, $\mathbf{\Sigma}_{\zeta\zeta}(\alpha)$ is not a covariance and the least eigenvalue is strictly increasing in α. The least eigenvalues are plotted against α in Figure 6, page 17.

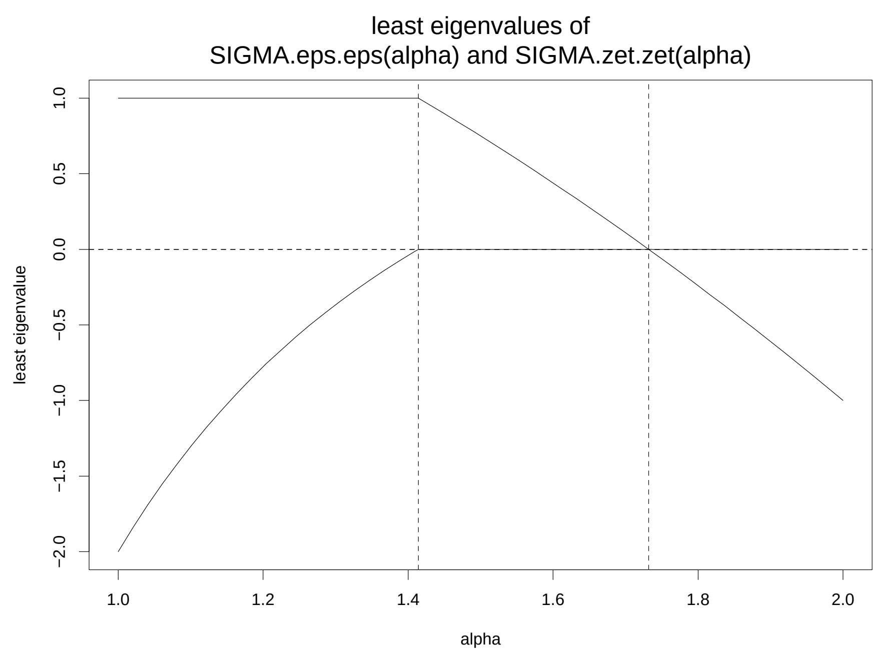

Figure 6: The least eigenvalues of $\mathbf{\Sigma}_{\epsilon\epsilon}(\alpha)$ (the nonincreasing function) and $\mathbf{\Sigma}_{\zeta\zeta}(\alpha)$ for the matrix (24), page 16.

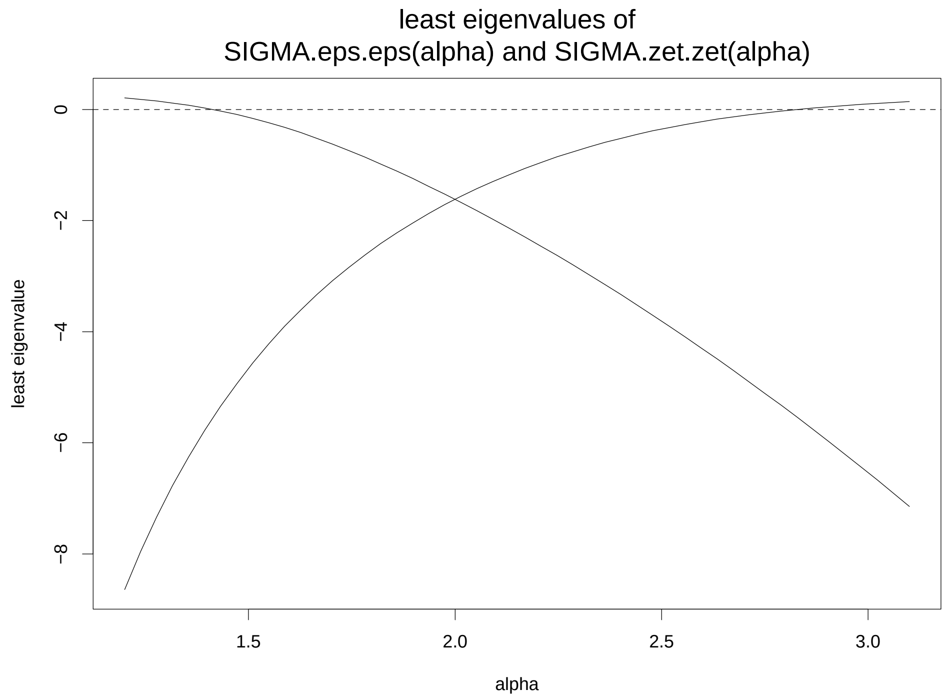

Figure 7: The least eigenvalues of $\mathbf{\Sigma}_{\epsilon\epsilon}(\alpha)$ (the decreasing function) and $\mathbf{\Sigma}_{\zeta\zeta}(\alpha)$ for the matrix at line (25), page 18. As we would expect, since this matrix fails to be positive semidefinite there is no α, or scale for the **X**-salience vector, by which the single latent model can parameterize it. This may be seen by the fact that there is no value of α for which both curves are greater than or equal to zero.

A matrix which cannot be parameterized. The following matrix,

$$\mathbf{\Sigma} = \begin{bmatrix} 2 & 1 & 2 & 2 \\ 1 & 1 & 2 & 2 \\ 2 & 2 & 2 & 1 \\ 2 & 2 & 1 & 1 \end{bmatrix}, \tag{25}$$

is not a variance; that is, it fails to be positive semidefinite. Its least eigenvalue is -1.62. The curves of least eigenvalues of $\mathbf{\Sigma}_{\epsilon\epsilon}(\alpha)$ and $\mathbf{\Sigma}_{\zeta\zeta}(\alpha)$ are plotted in Figure 7 on page 18, under the assumption that $p = q = 2$.

4 Discussion

Likelihood and the equivalence of model spaces. Three spaces of covariance matrices over the observed variables $\mathbf{X}$ and $\mathbf{Y}$ are of interest in the current work. They are:

1. Those corresponding to the rank-constraint model.
2. Those induced by the symmetric paired latent model.
3. Those induced by the symmetric single latent model.

It follows from definitions and from Equations (3) that Set 3 $\subset$ Set 2 $\subset$ Set 1. Theorem 2.1, however, implies that Set 1 $\subset$ Set 3. Hence Set 1 = Set 2 = Set 3, a fact which we state as the following corollary.

Corollary 4.1 *The sets of covariance matrices over the observed variables induced by the symmetric paired latent correlation model and the symmetric single latent model are equal to the set of covariance matrices belonging to the rank-one constraint model.*

Thus all single and paired latent parameterizations within an equivalence class have the same likelihood under the multivariate normal model for $\left(\mathbf{X}^T, \mathbf{Y}^T\right)^T$, and consequently there is no way using only data to distinguish between the three models. Furthermore it is well-known that the rank-one constraint model is covariance equivalent to reduced-rank regression (RRR). This fact is reviewed, for instance, in Wegelin [14]. Since maximum-likelihood estimation procedures are available for RRR (Anderson [1] [2] [3]), the problems of maximum-likelihood estimation for the paired and single symmetric latent models are solved, at least when the covariance matrix is invertible.

Within-block error tradeoffs in the single latent model. The feasible sets introduced at (13) and (20) characterize the degree to which a single latent or paired latent model is not identified.

Let us first consider the single latent model. If $\alpha_{\min} = \alpha_{\max}$ there is only one parameterization, say α^*. Since f and g, the functions used to define $\alpha_{\min}$ and $\alpha_{\max}$, are continuous, $f(\alpha^*) = g(\alpha^*) = 0$ and the unique parameterization yields singular within-block error variances for both blocks.

If the joint variance-covariance $\mathbf{\Sigma}$ is strictly positive definite, $\alpha_{\min} < \alpha_{\max}$. The converse is not true, however, because f or g may fail to be be strictly monotone. We have seen an example at (24) on page 16.

When $\alpha_{\min} < \alpha_{\max}$, the likelihood for the single latent model gives no information on what value of α within the feasible set should be chosen. The choice of α does however entail a decision as to which block of indicators is interpreted as measuring the latent variable more precisely. This is because of the way the within-block error variances vary with α in (11). The variances of the $\mathbf{X}$-block errors decrease linearly in α^2, those of the $\mathbf{Y}$-block in $\frac{1}{\alpha^2}$. Thus when a constraint distribution permits a within-block error covariance to be nonsingular, that error covariance can be made singular only by simultaneously

- reducing the variances of all the errors in the block, and
- increasing the variances of all the errors in the other block.

Thus α may be considered a "tradeoff parameter." When $\alpha = \alpha_{\min}$, $\mathbf{\Sigma}_{\zeta\zeta}$ is singular and consequently $\mathbf{Y}$ measures the latent $\boldsymbol{\eta}$ as closely as possible. At the same time the errors of the $\mathbf{X}$-block, $\mathbf{\Sigma}_{\epsilon\epsilon}$, have the greatest variance permitted by the constraint model. When $\alpha = \alpha_{\max}$ the reverse is true.

Correlation, identifiability, and tradeoffs in the paired latent correlation model. When $\alpha_{\min} = \alpha_{\max}$ the paired latent correlation model has a unique parameterization, just as the single latent model has. When $\alpha_{\min} < \alpha_{\max}$ this model, like the single latent model, is not identified. In this case, however, the feasible set lies in the plane, and we consequently have two tradeoffs, not one as with the single latent model.

The choice of the quantity ρ within the feasible interval $[\rho_{\min}, 1]$ entails a tradeoff between error variance on one hand and correlation on the other. When $\rho = \rho_{\min}$ the error variances for both blocks are at their minimum, and in fact the covariance matrices $\mathbf{\Sigma}_{\epsilon\epsilon}$ and $\mathbf{\Sigma}_{\zeta\zeta}$ are singular. When $\rho = 1$ the feasible values for α are exactly those for the single latent model, and the tradeoff described for that model applies. In particular, at least one of the error variances may be nonsingular.

Recall that $\rho_{\min} \equiv \frac{\alpha_{\min}}{\alpha_{\max}}$ is a constant determined by the constraint model, that is, by the joint population variance-covariance matrix $\mathbf{\Sigma}$. When this quantity is less than one we may choose to have latent variables perfectly correlated but poorly measured ($\rho = 1$); or latent variables measured with minimal error, in fact with a singular error distribution, but poorly correlated with each other ($\rho = \rho_{\min}$); or anything between these two extremes. The former entails an additional choice as to which block shall have greater error, as in the single latent model. If the latter choice, ($\rho = \rho_{\min}$), were adopted as a convention, and a sign convention were also adoped, the model would be identifiable.

Singular value decomposition. In two-block rank-one Mode A Partial Least Squares (PLS), an application of the singular value decomposition to the sample cross-covariance matrix, empirical saliences $\mathbf{u}$ and $\mathbf{v}$ are computed. These PLS saliences are scaled sample covariances between indicators and paired latent-variable scores, one for the $\mathbf{X}$-block and one for the $\mathbf{Y}$-block. The scores are computed as linear combinations of the variables for their respective blocks.

Although the PLS procedure depends on no statistical model, it is closely related to the family of paired latent models. This relationship is seen most easily when an equivalent paired latent model, the SVD paired latent model, is considered rather than the paired latent correlation model. The SVD paired latent model is defined and discussed in Section 5.3.

Provided the two largest singular values of $\mathbf{\Sigma}_{XY}$ are distinct and a sign convention has been adopted, it can easily be shown that the PLS saliences are consistent for the saliences or loadings of the SVD paired latent model. That is, as the number of observations approaches infinity, the values of the PLS saliences approach the loadings of the SVD paired latent model with probability one. This fact follows from the continuity of the singular value decomposition (Theorem 6.3.2, page 365 of Horn and Johnson [7]). Note that prior to Theorem 2.1 it was not known whether every population covariance matrix over $\left(\mathbf{X}^T, \mathbf{Y}^T\right)^T$ with $\operatorname{rank}(\mathbf{\Sigma}_{XY}) = 1$ could be interpreted as having arisen from a paired latent model. This has now been shown.

In PLS applications, vectors of scores on paired latent variables are computed as linear combinations of the indicators. The correlation between these variables has been estimated from the vectors of latent scores. This estimate of correlation is subject to the attenuation discussed by Spearman [12]. A traditional correction for attenuation however is not necessary.[1] Instead the lower bound for correlation shown in the current work may be used.

An example of two-block Mode A PLS, and of the interpretation of PLS saliences, may be seen in Streissguth et al. [13].

Factor models. Bollen ([4], pages 227ff.) distinguishes between confirmatory and exploratory factor models. He states that, in exploratory factor models, within-block errors or "measurement errors" are uncorrelated. In confirmatory factor models, on the other hand, these errors may be correlated. Thus single and paired latent models are closely related to confirmatory factor models. When they are identified they satisfy Bollen's definition, and may be considered confirmatory factor models.

A practical difference exists, however, between the current approach and the manner in which confirmatory factor models have customarily been treated. Although general statements of the confirmatory factor model family often place no *a priori* constraints on the within-block error covariance, few if any specific confirmatory factor models can be found in the literature with unconstrained within-block covariance. Reasons for this are both that such a model would be underidentified, and that it could be difficult to fit.

The current work deals with both difficulties. The degree to which the model is underidentified has been characterized. The model has been shown to be identified under the convention that $\rho = \rho_{\min}$. In addition the problem of

[1] There is an extensive literature on attenuation and on corrections for attenuation. Spearman's seminal article [11] was reprinted in 1987 [12]. Attenuation is mentioned by Kendall and Stuart [8], page 327, and by Fisher and van Belle [5], page 385. Lord and Novick provide a mathematical justification for the correction for attenuation [9]. Muchinsky reviews the issues and controversies surrounding disattenuation, including alternate formulas [10]. Zimmerman and Williams use simulation to investigate the properties of the disattenuated correlation under various conditions [15].

fitting the model by maximum likelihood has been transformed into the well-studied problem of fitting a reduced-rank regression model.

5 Appendix

5.1 Lemmas

To prove Theorem 2.1 we require the following lemmas.

Lemma 5.1 *Let* $\mathbf{A}$ *and* $\mathbf{C}$ *be symmetric matrices of the same dimension,* $\mathbf{C}$ *positive semidefinite. Let* $h : [0, \infty) \mapsto I\!R$ *be defined by*

$$h(\alpha) \quad = \quad \textit{the smallest eigenvalue of } (\mathbf{A} - \alpha\mathbf{C}) \; .$$

Then

1. *The function* h *is monotone nonincreasing. If* $\mathbf{C}$ *is strictly positive definite, the function is strictly monotone decreasing.*
2. $\lim_{\alpha \downarrow 0} h(\alpha) = h(0)$.
3. *If* $\mathbf{C}$ *has at least one positive eigenvalue,* $\lim_{\alpha \uparrow \infty} h(\alpha) = -\infty$.

Proof. Let $\mathbf{z}(\alpha)$ be the eigenvector belonging to the smallest eigenvalue of $(\mathbf{A} - \alpha\mathbf{C})$, without loss of generality let $||\mathbf{z}(\alpha)|| = 1$, and recall that, with this convention, $\mathbf{z}(\alpha)^T (\mathbf{A} - \alpha\mathbf{C}) \mathbf{z}(\alpha)$ equals the smallest eigenvalue.

Part 1. Let $\alpha < \beta$.

$$\begin{aligned}
h(\alpha) &= \mathbf{z}(\alpha)^T (\mathbf{A} - \alpha\mathbf{C}) \mathbf{z}(\alpha) \\
&= \mathbf{z}(\alpha)^T \mathbf{A}\mathbf{z}(\alpha) - \alpha\mathbf{z}(\alpha)^T \mathbf{C}\mathbf{z}(\alpha) \\
&\geq \mathbf{z}(\alpha)^T \mathbf{A}\mathbf{z}(\alpha) - \beta\mathbf{z}(\alpha)^T \mathbf{C}\mathbf{z}(\alpha) \qquad (26) \\
&= \mathbf{z}(\alpha)^T (\mathbf{A} - \beta\mathbf{C}) \mathbf{z}(\alpha) \\
&\geq \mathbf{z}(\beta)^T (\mathbf{A} - \beta\mathbf{C}) \mathbf{z}(\beta) \qquad (27) \\
&= h(\beta) \; .
\end{aligned}$$

At line (26) the inequality is not strict because possibly $\mathbf{z}(\alpha)^T \mathbf{C}\mathbf{z}(\alpha) = 0$. If $\mathbf{C}$ is strictly positive definite, the inequality at this line is strict and hence h is strictly decreasing. The inequality at line (27) occurs because $\mathbf{z}(\beta)$, by definition, minimizes the quadratic form.

Part 2. This is a consequence of a well-known theorem regarding the eigenvalues of a diagonalizable matrix under perturbation. See, for example, Horn and Johnson [7], Theorem 6.3.2, page 365. An ad-hoc proof of the continuity of h at 0 is included here as an exercise.

$$\begin{aligned}
h(\alpha) &\equiv \min_{||\mathbf{x}||=1} \left(\mathbf{x}^T \mathbf{A}\mathbf{x} - \alpha\mathbf{x}^T \mathbf{C}\mathbf{x} \right) \qquad (28) \\
&\geq \min_{||\mathbf{x}||=1} \mathbf{x}^T \mathbf{A}\mathbf{x} - \max_{||\mathbf{y}||=1} \left(\alpha\mathbf{y}^T \mathbf{C}\mathbf{y} \right) \; , \qquad (29)
\end{aligned}$$

since in line (28) the feasible set is a subset of the feasible set in line (29). Letting $\alpha \downarrow 0$ we see that the limit is bounded below by $h(0)$.

$$\begin{aligned} h(\alpha) &= \mathbf{z}(\alpha)^T (\mathbf{A} - \alpha\mathbf{C})\, \mathbf{z}(\alpha) \\ &\leq \mathbf{z}(0)^T (\mathbf{A} - \alpha\mathbf{C})\, \mathbf{z}(0) \\ &= h(0) - \alpha\mathbf{z}(0)^T \mathbf{C}\mathbf{z}(0) \ . \end{aligned}$$

Thus the limit is also bounded above by $h(0)$.

Part 3. Let $\mathbf{y}$ be an eigenvector corresponding to a positive eigenvalue of $\mathbf{C}$.

$$\min_{||\mathbf{x}||=1} \mathbf{x}^T (\mathbf{A} - \alpha\mathbf{C})\, \mathbf{x} \quad \leq \quad \mathbf{y}^T \mathbf{A}\mathbf{y} - \alpha\mathbf{y}^T \mathbf{C}\mathbf{y} \ .$$

The first term is constant. The second term approaches negative infinity as α approaches infinity.

Lemma 5.2 *Let* $\mathbf{x}$ *be a* p*-vector,* $\mathbf{y}$ *a* q*-vector. Let* $\mathbf{U}$ *be* $p \times R$*,* $\mathbf{V}$ $q \times R$*, let* $\mathbf{W} = \begin{bmatrix} \mathbf{U} \\ \mathbf{V} \end{bmatrix}$*,* $\mathbf{Q} = \mathbf{W}\mathbf{W}^T$*. Let the entries in these matrices and vectors be real, and consider the quadratic form*

$$Z(t) \quad = \quad \begin{bmatrix} \mathbf{x}^T, t\mathbf{y}^T \end{bmatrix} \quad \mathbf{Q} \quad \begin{bmatrix} \mathbf{x} \\ t\mathbf{y} \end{bmatrix} \ .$$

Then there is a real t *such that* $Z(t) = 0$ *if and only if one or more of the following conditions holds:*

$$\begin{aligned} \mathbf{x}^T \mathbf{U} &= \mathbf{0} \ , \\ \mathbf{y}^T \mathbf{V} &= \mathbf{0} \ , \\ \mathbf{x}^T \mathbf{U} &\propto \mathbf{y}^T \mathbf{V} \ . \end{aligned}$$

Furthermore the real solution, if it exists, is unique.

Proof.

$$\begin{aligned} Z(t) &= \begin{bmatrix} \mathbf{x}^T, t\mathbf{y}^T \end{bmatrix} \begin{bmatrix} \mathbf{U} \\ \mathbf{V} \end{bmatrix} \begin{bmatrix} \mathbf{U}^T, \mathbf{V}^T \end{bmatrix} \begin{bmatrix} \mathbf{x} \\ t\mathbf{y} \end{bmatrix} \\ &= \left(\mathbf{x}^T \mathbf{U} + t\mathbf{y}^T \mathbf{V}\right) \left(\mathbf{U}^T \mathbf{x} + t\mathbf{V}^T \mathbf{y}\right) \\ &= \left(\mathbf{x}^T \mathbf{U}\mathbf{U}^T \mathbf{x}\right) + 2t \left(\mathbf{x}^T \mathbf{U}\mathbf{V}^T \mathbf{y}\right) + t^2 \left(\mathbf{y}^T \mathbf{V}\mathbf{V}^T \mathbf{y}\right) \ . \end{aligned}$$

This is quadratic in t. Let $\mathbf{z} = \mathbf{x}^T \mathbf{U}$ and $\mathbf{w} = \mathbf{y}^T \mathbf{V}$, and let θ be the angle between $\mathbf{z}$ and $\mathbf{w}$. Then $Z(t)$ is of the form

$$Z(t) \quad = \quad at^2 + bt + c \ ,$$

where

$$\begin{array}{rclclcl} a & = & \mathbf{y}^T\mathbf{V}\mathbf{V}^T\mathbf{y} & = & \mathbf{w}^T\mathbf{w} & = & ||\mathbf{w}||^2 \ , \\ b & = & 2\mathbf{x}^T\mathbf{U}\mathbf{V}^T\mathbf{y} & = & 2\mathbf{z}^T\mathbf{w} & = & 2\ ||\mathbf{z}||\ ||\mathbf{w}||\ \cos\theta \ , \text{ and} \\ c & = & \mathbf{x}^T\mathbf{U}\mathbf{U}^T\mathbf{x} & = & \mathbf{z}^T\mathbf{z} & = & ||\mathbf{z}||^2 \ . \end{array}$$

If the discriminant is nonnegative, there is a real root t such that $Z(t) = 0$; if the discriminant is zero, the root is unique.

$$\begin{aligned} \frac{b^2 - 4ac}{4} &= ||\mathbf{z}||^2\ ||\mathbf{w}||^2\ \cos^2\theta - ||\mathbf{z}||^2\ ||\mathbf{w}||^2 \\ &= ||\mathbf{z}||^2\ ||\mathbf{w}||^2 \left(\cos^2\theta - 1\right) \ . \end{aligned}$$

This value is real and nonpositive. It is zero if and only if at least one of the conditions holds which are stated in the lemma. □

Corollary 5.3 *If $R = 1$, there is a unique t such that $Z(t) = 0$.*

Proof. Apply Lemma 5.2, and notice that in this case $\mathbf{x}^T\mathbf{U}$ and $\mathbf{y}^T\mathbf{V}$ are scalars. □

Lemma 5.4 *Let*

$$\mathbf{\Sigma} = \begin{bmatrix} \mathbf{A} & \mathbf{C} \\ \mathbf{C}^T & \mathbf{B} \end{bmatrix} ,$$

where $\mathbf{\Sigma}$ is symmetric positive semidefinite, $\mathbf{A}$ and $\mathbf{B}$ are respectively $p \times p$ and $q \times q$, and $\mathbf{C}$ is of rank one. Let $\mathbf{u}$ and $\mathbf{v}$ be p- and q-vectors satisfying

$$\mathbf{C} = \mathbf{u}\mathbf{v}^T \ .$$

Define

$$\begin{aligned} \mathbf{A}^* &= \mathbf{A} - \mathbf{u}\mathbf{u}^T \ , \\ \mathbf{B}^* &= \mathbf{B} - \mathbf{v}\mathbf{v}^T \ . \end{aligned}$$

Then at least one of $\mathbf{A}^*$ and $\mathbf{B}^*$ is positive semidefinite. Furthermore, if $\mathbf{\Sigma}$ is positive definite, at least one of $\mathbf{A}^*$ and $\mathbf{B}^*$ is positive definite.

Proof. Let $\mathbf{x}$ and $\mathbf{y}$ be the eigenvectors of $\mathbf{A}^*$ and $\mathbf{B}^*$ corresponding to their smallest eigenvalues, δ and ϵ, so that

$$\mathbf{A}^*\mathbf{x} = \delta\mathbf{x} \quad \text{and} \quad \mathbf{B}^*\mathbf{y} = \epsilon\mathbf{y} \ .$$

Let

$$\mathbf{w} = \begin{bmatrix} \mathbf{u} \\ \mathbf{v} \end{bmatrix} , \quad \mathbf{Q} = \mathbf{w}\mathbf{w}^T , \quad \mathbf{E} = \begin{bmatrix} \mathbf{A}^* & \mathbf{0} \\ \mathbf{0} & \mathbf{B}^* \end{bmatrix} ,$$

so that

$$\mathbf{\Sigma} \;=\; \mathbf{Q} + \mathbf{E}\ .$$

For real t, consider the following quadratic form:

$$\begin{aligned}\left[\mathbf{x}^T, t\mathbf{y}^T\right] \quad \mathbf{\Sigma} \begin{bmatrix} \mathbf{x} \\ t\mathbf{y} \end{bmatrix} &= \left[\mathbf{x}^T, t\mathbf{y}^T\right] \quad \mathbf{Q} \begin{bmatrix} \mathbf{x} \\ t\mathbf{y} \end{bmatrix} + \mathbf{x}^T\mathbf{A}^*\mathbf{x} + t^2\mathbf{y}^T\mathbf{B}^*\mathbf{y} \\ &= \left[\mathbf{x}^T, t\mathbf{y}^T\right] \quad \mathbf{Q} \begin{bmatrix} \mathbf{x} \\ t\mathbf{y} \end{bmatrix} + \delta + t^2\epsilon\ .\end{aligned}$$

By Corollary 5.3, there is a unique real t such that the first term is zero. Thus

$$\max(\delta, \epsilon) < 0 \quad \Rightarrow \quad \mathbf{\Sigma} \text{ has a negative eigenvalue, and}$$

$$\max(\delta, \epsilon) \le 0 \quad \Rightarrow \quad \mathbf{\Sigma} \text{ has a nonpositive eigenvalue.}$$

By the contrapositive, it follows that, if $\mathbf{\Sigma}$ is strictly positive definite, then at least one of $\mathbf{A}^*$ and $\mathbf{B}^*$ is strictly positive definite; and if $\mathbf{\Sigma}$ is positive semidefinite, then at least one of $\mathbf{A}^*$ and $\mathbf{B}^*$ is positive semidefinite. □

5.2 Facts related to generalized eigenvalues

The following facts are used in Section 2.2.

Proposition 5.5 *Let* $\mathbf{A}$ *and* $\mathbf{C}$ *be* $n \times n$ *matrices. If* $\lambda \neq 0$*, then*

$$|\mathbf{A} - \lambda\mathbf{C}| \;=\; 0 \;\Leftrightarrow\; \left|\mathbf{C} - \frac{1}{\lambda}\mathbf{A}\right| \;=\; 0\ .$$

Proof. Multiply both sides of the expression on the left by $\left(-\frac{1}{\lambda}\right)^n$. □

Proposition 5.6 *Let* $\mathbf{A}$ *be nonsingular. Then any generalized eigenvalue* λ *satisfying* $|\mathbf{C} - \lambda\mathbf{A}| = 0$ *is an eigenvalue of* $\mathbf{A}^{-1}\mathbf{C}$.

Proof. For some nonzero $\mathbf{x}$ we have

$$\begin{aligned}\mathbf{0} &= \mathbf{C}\mathbf{x} - \lambda\mathbf{A}\mathbf{x} \\ &= \mathbf{A}^{-1}\mathbf{C}\mathbf{x} - \lambda\mathbf{x}\ .\end{aligned}$$

□

5.3 SVD paired latent models

The rank-one **SVD paired latent model** is equivalent to the rank-one paired latent correlation model specified at (2). The rank-one SVD paired latent model is the set of distributions over the latent variables $\boldsymbol{\xi}$ and $\boldsymbol{\omega}$, the observed variables $\mathbf{x}$ and $\mathbf{y}$, and the errors $\boldsymbol{\epsilon}$ and $\boldsymbol{\zeta}$, specified as follows.

$$\left.\begin{array}{l}\left.\begin{array}{rcl}\mathbf{x} & = & \mathbf{u}\boldsymbol{\xi} + \boldsymbol{\epsilon} \ , \\ \mathbf{y} & = & \mathbf{v}\boldsymbol{\omega} + \boldsymbol{\zeta} \ , \end{array}\right\} \text{where} \\ \mathbf{Var}\begin{bmatrix} \boldsymbol{\xi} \\ \boldsymbol{\omega} \end{bmatrix} = \begin{bmatrix} \phi & d \\ d & \psi \end{bmatrix} \ , \\ \mathbf{Var}(\boldsymbol{\epsilon}) = \boldsymbol{\Sigma}_{\epsilon\epsilon} \ , \\ \mathbf{Var}(\boldsymbol{\zeta}) = \boldsymbol{\Sigma}_{\zeta\zeta} \ , \\ \boldsymbol{\epsilon} \perp\!\!\!\perp \begin{bmatrix} \boldsymbol{\xi} \\ \boldsymbol{\omega} \end{bmatrix} , \quad \boldsymbol{\epsilon} \perp\!\!\!\perp \boldsymbol{\zeta} \ , \quad \begin{bmatrix} \boldsymbol{\xi} \\ \boldsymbol{\omega} \end{bmatrix} \perp\!\!\!\perp \boldsymbol{\zeta} \ , \\ \mathbf{u} \in I\!\!R^p \ , \quad \mathbf{v} \in I\!\!R^q \ , \quad ||\mathbf{u}|| = ||\mathbf{v}|| = 1 \ . \end{array}\right\} \tag{30}$$

Thus the parameters of the SVD paired latent model are ϕ, ψ, d, $\mathbf{u}$, $\mathbf{v}$, $\boldsymbol{\Sigma}_{\epsilon\epsilon}$, and $\boldsymbol{\Sigma}_{\zeta\zeta}$, subject to the following constraints:

$$||\mathbf{u}|| = ||\mathbf{v}|| = 1 \ , \tag{31}$$

$$\boldsymbol{\Sigma}_{\epsilon\epsilon} \text{ positive semidefinite,} \tag{32}$$

$$\boldsymbol{\Sigma}_{\zeta\zeta} \text{ positive semidefinite,} \tag{33}$$

$$\phi\psi \geq d^2 \ . \tag{34}$$

The parameters of the SVD paired latent model may be partitioned into those which govern cross-covariance,

$$\mathbf{u}, \mathbf{v}, d \ ,$$

and those which govern within-block covariance,

$$\phi, \psi, \boldsymbol{\Sigma}_{\epsilon\epsilon}, \boldsymbol{\Sigma}_{\zeta\zeta} \ .$$

Such a partition is not possible in the paired latent correlation model. Note that d is a covariance, not a correlation. The correlation between the latents, ρ, is a function of the parameters:

$$\rho = \frac{d}{\sqrt{\phi\psi}} \ . \tag{35}$$

A rank-one constraint model is mapped to an SVD paired latent model as follows. Using the singular value decomposition, determine d, $\mathbf{u}$, and $\mathbf{v}$ by

$$\boldsymbol{\Sigma}_{XY} = d\mathbf{u}\mathbf{v}^T \ , \tag{36}$$

where, as we have noted, a sign convention is necessary to make these parameters identifiable. Determine the constants $\phi_{\max}$ and $\psi_{\max}$ by

$$\begin{aligned} \phi_{\max} &\equiv \max\left\{\phi : \left(\mathbf{\Sigma}_{XX} - \phi \mathbf{u}\mathbf{u}^T\right) \text{ is positive semidefinite}\right\} , \\ \psi_{\max} &\equiv \max\left\{\psi : \left(\mathbf{\Sigma}_{YY} - \psi \mathbf{v}\mathbf{v}^T\right) \text{ is positive semidefinite}\right\} . \end{aligned} \tag{37}$$

By Lemma 5.1, $\phi_{\max}$ and $\psi_{\max}$ exist. For future reference we recall line (14), page 7, noting thereby that

$$\phi_{\max} = \alpha^2_{\max} , \quad \psi_{\max} = \frac{d^2}{\alpha^2_{\min}} . \tag{38}$$

Then each point in the two-dimensional feasible set,

$$\left\{(\phi, \psi) : \phi \leq \phi_{\max} , \psi \leq \psi_{\max} , \phi\psi \geq d^2\right\} , \tag{39}$$

represents a SVD parameterization of the rank constraint model. That this set is nonempty is a consequence of Theorem 2.1. For any (ϕ, ψ) in the feasible set, the remaining parameters are defined by

$$\begin{aligned} \mathbf{\Sigma}_{\epsilon\epsilon}(\phi) &= \mathbf{\Sigma}_{XX} - \phi \mathbf{u}\mathbf{u}^T , \\ \mathbf{\Sigma}_{\zeta\zeta}(\psi) &= \mathbf{\Sigma}_{YY} - \psi \mathbf{v}\mathbf{v}^T . \end{aligned} \tag{40}$$

To see that this parameterization is correct, it suffices to note that the following requirements are met.

1. The parameter values defined recover $\mathbf{\Sigma}$—that is, this parameterization maps into the constraint model with which we started. This follows immediately from (36) and (40).

2. The constraints at (39) are necessary and sufficient for the parameters to be feasible—that is, for $\mathbf{\Sigma}_{\epsilon\epsilon}$, $\mathbf{\Sigma}_{\zeta\zeta}$, and $\begin{bmatrix} \phi & d \\ d & \psi \end{bmatrix}$ to be positive semidefinite. This follows immediately from the definitions of $\phi_{\max}$ and $\psi_{\max}$, and the condition $\phi\psi > d^2$ in the definition of the feasible set.

The values of ϕ and ψ corresponding to the single latent model form the lower left boundary of the feasible set, which is the intersection of the curve

$$\psi = \frac{d^2}{\phi}$$

with the rectangle $[0, \phi_{\max}] \times [0, \psi_{\max}]$. For constraint models such that $\alpha_{\max} = \alpha_{\min}$, the feasible set degenerates to the single point $(\phi_{\max}, \psi_{\max})$.

Define $\phi_{\min}$ and $\psi_{\min}$ by

$$\phi_{\min} \equiv \frac{d^2}{\psi_{\max}} , \quad \psi_{\min} \equiv \frac{d^2}{\phi_{\max}} . \tag{41}$$

The constraints

$$\phi \geq \phi_{\min} \ , \quad \psi \geq \psi_{\min}$$

are implicit in the definition of the feasible set, since both ϕ and ψ have maximum values, and $\phi\psi \geq d^2$. At any point where ϕ or ψ attains its maximum the covariance of the latents is singular. The minimum value for ϕ is attained only when ψ attains its maximum, and vice versa.

The value $(\phi_{\max}, \psi_{\max})$ entails a nonsingular variance for $(\boldsymbol{\xi}, \boldsymbol{\omega})^T$ whenever the feasible set is not degenerate. In all cases, however, $(\phi_{\max}, \psi_{\max})$ entails singular within-block error variances for both blocks. The minimum feasible correlation is attained at this point, and defined by

$$\rho_{\min} \quad \equiv \quad \frac{d}{\sqrt{\phi_{\max}\psi_{\max}}} \ .$$

Bijection between parameterizations. It was stated on page 26 that the SVD paired latent model and the paired latent correlation model are equivalent. This may be seen by the fact that a bijection exists between the feasible sets for the two parameterizations, as defined at (20) and at (39). Given a point (α, ρ) in the feasible set for the paired latent correlation model, we obtain a point in the feasible set for the SVD paired latent model by

$$\phi \quad = \quad \alpha^2 \ , \quad \psi \quad = \quad \frac{d^2}{\alpha^2 \rho^2} \ .$$

Given a point (ϕ, ψ) in the feasible set for the SVD paired latent model, we obtain a point in the feasible set for the paired latent correlation model by

$$\alpha \quad = \quad \sqrt{\phi} \ , \quad \rho \quad = \quad \frac{d}{\sqrt{\phi\psi}} \ .$$

Example. Let us return to the constraint model (22) on page 11, and compare its SVD-model feasible set with its paired-correlation-model feasible set. Recall that

$$d \quad = \quad \sqrt{\frac{35}{2}} \ , \quad \alpha_{\min} \quad = \quad \frac{1}{30\sqrt{2030}} \ , \quad \alpha_{\max} \quad = \quad \sqrt{7} \ .$$

Comparing (14) on page 7 with (37) on page 27 we see that the constraints defining the SVD-model feasible set are obtained by

$$\phi_{\max} \quad = \quad \alpha^2_{\max} \quad = \quad 7 \ ,$$

$$\psi_{\max} \quad = \quad \frac{d^2}{\alpha^2_{\min}} \quad = \quad \frac{225}{29} \quad \approx \quad 7.76 \ .$$

Then we also have

$$\phi_{\min} = \frac{d^2}{\psi_{\max}} = \frac{203}{90} \approx 2.26 \,,$$

$$\psi_{\min} = \frac{d^2}{\phi_{\max}} = \frac{5}{2} = 2.5 \,,$$

$$\rho_{\min} = \frac{d}{\sqrt{\phi_{\max}\psi_{\max}}} = \frac{\sqrt{290}}{30} \approx 0.568 \,.$$

The feasible set is plotted in Figure 8 on page 30. The point where $\rho = \rho_{\min}$ and consequently $\mathbf{\Sigma}_{\epsilon\epsilon}(\phi)$ and $\mathbf{\Sigma}_{\zeta\zeta}(\psi)$ are singular is in the upper right corner of the feasible set. In Figure 4 on page 14, the level set for the paired latent correlation model, by contrast, $\rho = \rho_{\min}$ at the leftmost point of the feasible set. The points corresponding to the single latent model ($\rho = 1$) in this Figure are on the lower left curved boundary, whereas in Figure 4 they are on the right boundary.

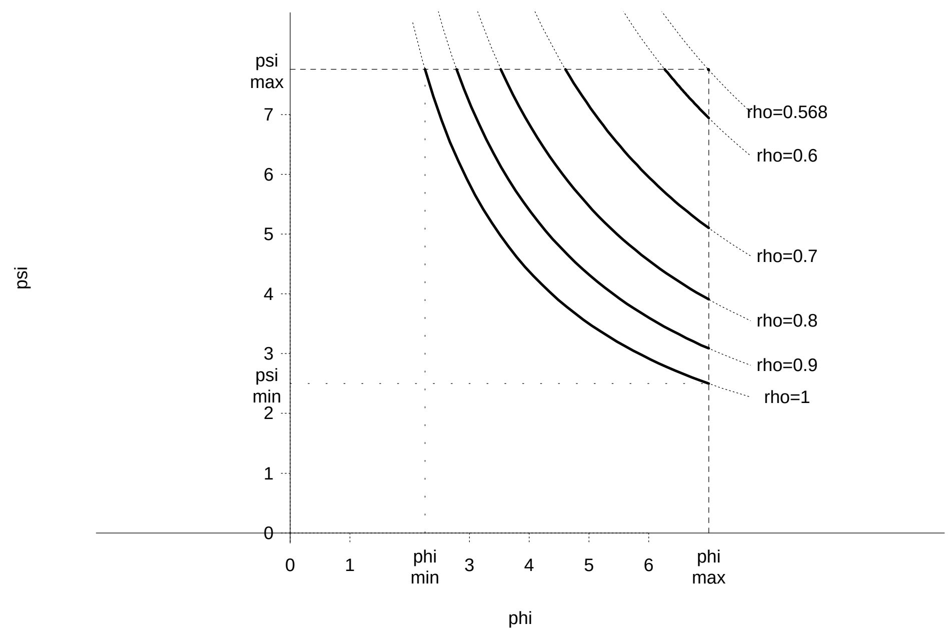

Figure 8: Feasible set for the SVD parameterization of the constraint model (22) on page 11. The SVD paired latent model is defined in Section 5.3. A bijection exists between this feasible set and the feasible set plotted in Figure 4 on page 14, representing the paired latent correlation parameterization of the same constraint model. Level sets of ρ lie on the curves $\psi = \dfrac{d^2}{\rho^2 \phi}$. Feasible values for ρ are in the closed interval $\left[\dfrac{\sqrt{290}}{30}, 1\right]$. At $\rho = \dfrac{\sqrt{290}}{30} \approx 0.568$, the level set is the point $\left(7, \dfrac{225}{29}\right)$, that is, the upper right corner of the broken rectangle. A selection of level sets for ρ are plotted as solid curves. Feasible values for ϕ and ψ lie inside the broken rectangle and on or to the right of the $\rho = 1$ curve.

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