THE ROLE OF REVERSALS IN ORDER-RESTRICTED INFERENCE

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ABSTRACT. An order-restricted (OR) statistical model can be viewed as one whose parameter space is a closed convex cone C in a Euclidean space. Order-restricted likelihood ratio tests and maximum likelihood estimators have been criticised on the grounds that they may violate the cone-order monotonicity (COM) property relative to C in the common case that C is obtuse; instead they may "reverse" the cone-order. It is argued here, however, that the COM property is an inappropriate requirement for OR tests and estimates.

Key Words: Order-restricted inference, likelihood ratio test, maximum likelihood estimator, cone-order monotonicity, reversals.

1. Cone-order monotonicity in order-restricted inference.

An order-restricted (OR) statistical model $\{P_\theta \mid \theta \in C\}$ can be considered to be one whose parameter space C is a closed convex cone in a p-dimensional Euclidean space $\mathbb{R}^p$ – cf. Robertson *et al* (1988). Here p is assumed to be finite, although infinite-dimensional OR models also occur. If the statistical model $\{P_\theta\}$ constitutes an exponential family, after reduction by sufficiency the sample space can also be taken to be a subset of $\mathbb{R}^p$, which we assume here.

The cone C may be pointed (i.e., contains no non-zero linear subspace) or non-pointed: a non-pointed cone may be thought of as an infinite wedge, or book, whose "spine" L is a proper linear subspace of $\mathbb{R}^p$. For example, the nonnegative orthant in $\mathbb{R}^p$ is a pointed cone while the *simple-order* cone

$$C_s \equiv C_s^{(p)} \equiv \{(\theta_1, \ldots, \theta_p) \in \mathbf{R}^p \mid \theta_1 \leq \cdots \leq \theta_p\} \tag{1}$$

is non-pointed since it contains the one-dimensional line

$$L_s \equiv L_s^{(p)} \equiv \{(\theta_1, \ldots, \theta_p) \mid \theta_1 = \cdots = \theta_p\}. \tag{2}$$

Any non-pointed cone C can be uniquely represented as the product

$$C = L \times C^* \tag{3}$$

of its spine L (formally, the maximal linear subspace contained in C) and the pointed cone C^* obtained by projecting C onto the orthogonal complement $L^\perp$. If C is pointed then (3) holds trivially with $L = \{0\}$ and $C^* = C$. We call the cone C *obtuse* if the maximal angle between any two extreme rays in C^* exceeds 90°, otherwise C is called *acute*.

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The simple order cone C_s is acute, but most other cones occurring in OR models are obtuse. In particular, the *tree-order* cone C_t defined by

$$C_t \equiv C_t^{(k)} \equiv \{(\mu_0, \mu_1, \ldots, \mu_k) \in \mathbb{R}^{k+1} \mid \mu_0 \leq \mu_i,\ i = 1, \ldots, k\} \tag{4}$$

is obtuse. Here $p = k + 1$ and $C_t^{(k)}$ is a non-pointed obtuse cone with spine

$$L_t \equiv L_t^{(k)} \equiv \{(\mu_0, , \mu_1, \ldots, \mu_k) \mid \mu_0 = \mu_1 = \cdots = \mu_k\}, \tag{5}$$

again a one-dimensional line. The tree-order cone arises naturally when considering the problem of comparing several treatments to a control.

In a recent series of papers, Cohen and Sackrowitz [CS] assert that the order-restricted likelihood ratio test (ORLRT) and order-restricted maximum likelihood estimator (ORMLE) may be unsatisfactory for many OR inference problems (cf. CS (1998, 2000, 2002a, 2002b), Cohen *et al.* (2000); also Logan (2003)). This assertion is based on their view that a property they call *cone-order monotonicity* (COM) with respect to the cone C is "intuitively desirable" for tests and estimators. They demonstrate that when C is obtuse, the ORLRT and ORMLE may fail to satisfy COM, which they call a "reversal" of the cone order determined by C, hence conclude the "undesirability" of these procedures. They proceed to develop alternative tests and estimators which are COM and therefore apparently are preferable to the ORLRT and ORMLE for obtuse cones. Here we reexamine the role of reversals in OR testing and estimation problems.

2. The role of reversals for OR tests.

The most common OR hypothesis-testing problem is that of testing

$$H_0 : \theta \in L \qquad \text{vs.} \qquad H : \theta \in C \setminus L. \tag{6}$$

With $C = C_s$, for example, this becomes the standard problem of testing homogeneity $\theta_1 = \cdots = \theta_p$ against the ordered alternative $\theta_1 \leq \cdots \leq \theta_p$. For the general problem (6), the COM condition imposes the requirement that the test function ϕ should be monotone w.r.to the *cone-order* $\preceq_C$ defined on $\mathbb{R}^p$ as follows: $x \preceq_C y$ iff $y \in x + C$ (see Figure 1). Thus, COM requires that $\phi(x) \leq \phi(y)$ whenever $x \preceq_C y$. CS assert that since the sample point y lies "deeper" inside the cone C than does x, it is "intuitively evident" that the former provides more evidence in favour of the alternative hypothesis, so any "reasonable" test that rejects H_0 for x should also reject H_0 for y.

Assuming independent normal distributions with unknown means and equal known variances for simplicity, Perlman and Wu (2002) demonstrate that the CS intuition behind the COM property for the testing problem (6) is inapplicable when C is obtuse. In this case, *in relation to the null hypothesis*, $x \preceq_C y$ need *not* imply that y lies "deeper" inside C than does x. This can be seen in Figure 2, where $x \preceq_C y \equiv 0$ but $y \equiv 0$ is actually closer to (in fact, coincides with!) the null hypothesis (here, the vertex of the cone C) than is x. Thus, for an obtuse cone the "reversing" property of a test for (6) is not detrimental to its performance but is in fact entirely appropriate for certain sample configurations. In particular, Perlman and Wu argue that the ORLRT for (6) is an entirely reasonable test procedure for any cone C, whether obtuse or acute.

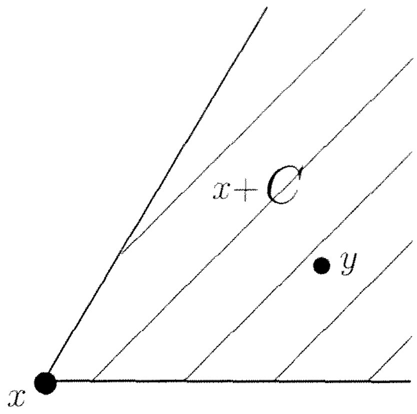

FIGURE 1. The cone order $\preceq_C$: $x \preceq_C y$ iff $y \in x + C$.

3. THE ROLE OF REVERSALS FOR OR ESTIMATORS.

For the problem of estimating θ in the OR model $\{P_\theta \mid \theta \in C\}$, an estimator $\hat{\theta}(x)$ is called COM if $x \preceq_C y$ implies $\hat{\theta}(x) \preceq_C \hat{\theta}(y)$. The role of "reversals" of the cone ordering is somewhat more subtle for estimation than for testing. Like the ORLRT, the ORMLE has been criticised on the grounds that it may "reverse" the cone ordering (again, this can occur only when C is obtuse) and thereby yield misleading estimates. Here we refute this criticism by arguing that these estimates are not misleading but in fact conform better to the order restrictions than do the alternative estimates of CS, and that the COM monotonicity requirement imposes an unreasonable restriction of another kind.

CS (2002a, p.91) present the following example to justify their contention that the COM property is "intuitively desirable" and hence that the ORMLE, which may reverse the cone order, is unsatisfactory. Consider the tree-order estimation problem for $k = 2$ based on the independent normal random variables $X_i \sim N(\mu_i, \sigma^2)$, $i = 0, 1, 2$, with σ^2 known. It is desired to estimate (μ_0, μ_1, μ_2) subject to the tree-order constraints $\mu_0 \le \mu_1$ and $\mu_0 \le \mu_2$.

For specificity, CS assume that μ_1 and μ_2 represent the mean responses to Drugs 1 and 2, respectively, while μ_0 represents the mean response to a placebo. Suppose that $(X_0, X_1, X_2) = (50, 50, -10)$. The ORMLE, which under normality is just the projection of the sample vector (X_0, X_1, X_2) onto the tree-order cone $C_t^{(2)}$, is given by

$$(\hat{\mu}_0, \hat{\mu}_1, \hat{\mu}_2) = (20, 50, 20), \tag{7}$$

while CS recommend an alternative estimator that here yields the estimates

$$(\tilde{\mu}_0, \tilde{\mu}_1, \tilde{\mu}_2) = (30, 30, 30). \tag{8}$$

CS argue that the ORMLE misleadingly indicates that Drug 1 is superior to the placebo, while their intuition (apparently based on the unadjusted sample means $(50, 50, -10)$) dictates that neither drug is superior to the placebo. Their alternative estimator (8) agrees with their intuition and therefore, they conclude, must be preferable to the ORMLE.

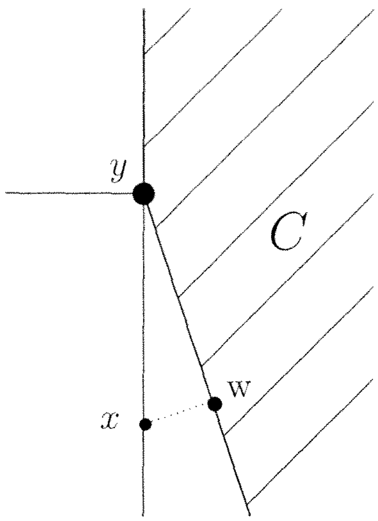

FIGURE 2. Example of an obtuse cone C, where COM is inappropriate.

This argument can be criticized, however, on the basis that it does not focus directly on the order restrictions. While the observed pair $(X_0, X_1) \equiv (50, 50)$ conforms to the assumed constraint $\mu_0 \leq \mu_1$, the pair $(X_0, X_2) \equiv (50, -10)$ does not conform to the corresponding constraint $\mu_0 \leq \mu_2$, hence should be adjusted. The ORMLE does this in the most parsimonious way, leading to the adjusted estimates $(\hat{\mu}_0, \hat{\mu}_2) = (20, 20)$, which preserve the sum $X_0 + X_2 = 40$. These estimates in conjunction with the unadjusted value $X_1 \equiv 50$ continue to satisfy the first constraint $\mu_0 \leq \mu_1$, so no further adjustment is necessary, yielding the complete ORMLE (7).

We acknowledge that this argument is heuristic, so, like that of CS, is also based on "intuition". We believe, however, that our intuition conforms more closely to the essence of the OR model.

A more general shortcoming of the COM requirement for OR estimation can be demonstrated by a two-dimensional obtuse cone C as in Figure 2. For simplicity, assume that $X \equiv (X_1, X_2) \sim N(\mu_1, \sigma^2) \otimes N(\mu_2, \sigma^2)$ with σ^2 known and consider the OR model given by $\mu \equiv (\mu_1, \mu_2) \in C$ with C as in Figure 2. To satisfy the OR condition, any estimator $\tilde{\mu}$ must satisfy $\tilde{\mu}(x) \in C$ for all sample points $x \equiv (x_1, x_2)$, while by normality it is reasonable to require that $\tilde{\mu}(x) = x$ if $x \in C$, that is, if x itself satisfies the order restriction. If $\tilde{\mu}$ is also required to be COM, then (see Figure 2) $x \preceq_C y \equiv (0, 0)$ implies that $\tilde{\mu}(x) \preceq_C \tilde{\mu}(y) \equiv (0, 0)$ (since $(0, 0) \in C$), or equivalently $\tilde{\mu}(x) \in -C$. Since also $\tilde{\mu}(x) \in C$, necessarily $\tilde{\mu}(x) = (0, 0)$ *no matter how far below* $(0, 0)$ *the sample point* x *lies.*

For example, if $\sigma^2 = 1$, $x = (0, -5)$, and the slope of the lower boundary of the cone C is -10, then the estimate $\tilde{\mu}(x) \equiv (0, 0)$ lies 5σ away from the observation x, but there is a point $w \in C$ that lies less than $(1/2)\sigma$ away from the observation.

It is easy to see that such behavior also occurs for obtuse cones of dimension greater than two. We conclude that COM is an unreasonable requirement in OR estimation problems determined by obtuse cones.

4. Concluding remarks

The ORMLE has been criticised for a second reason, namely that at least one of its components (or linear combination of its components) may have substantially greater bias and mean-square error (MSE) than the corresponding component (or linear combination) of the unrestricted MLE, especially in high dimensions (cf. Lee (1988), Robertson *et al.* (1988), Hwang and Peddada (1994), Fernandez *et al.* (1999), Cohen and Sackrowitz (2002a)). This second criticism, which holds for both obtuse and acute cones, is more substantial than the COM-based criticism and cannot be dismissed.

By viewing a high-dimensional OR estimation problem as an estimation problem with many nuisance parameters, however, this behaviour of the ORMLE can be explained by the fact that information about the target parameter (e.g., μ_0 in the tree-order estimation problem) does not accumulate as rapidly as the number of nuisance parameters (e.g., $\mu_1, \ldots, \mu_k$ in the tree-order problem). As in the classical example of Neyman and Scott (1948, Example 2), either the ORMLE can be adjusted to reduce its bias and/or even achieve consistency, or else no consistent estimator exists, depending upon the rate of accumulation of information about the target parameter. These topics are addressed in Chaudhuri and Perlman (2003a,b).

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