1) Simplify the following. Do not use parentheses or negative exponents in the final answer.

a) $\log_{\sqrt{27}} 81$

**Solution (2):** Let $x = \log_{\sqrt{27}} 81$ then, by the definition of the logarithm, $(\sqrt{27})^x = 81$, or $(3^{3/2})^x = 3^4$, so $x = 8/3$.

b) $\log_{36}(1/\sqrt{6})$

**Solution (2):** Let $x = \log_{36}(1/\sqrt{6})$ then, by the definition of the logarithm, $(36)^x = 1/\sqrt{6}$, or $(6^2)^x = 6^{-1/2}$, so $x = -1/4$.

2) **Compound Growth** A population is growing according to the formula:

$$P(t) = 5 \times 10^6 e^{0.06t}$$

where $t$ is in years. Calculate the percentage growth per annum. How long does it take the population to increase by 50%.

**Solution (8):** Consider the proportional growth from time $t$ to time $t + 1$:

$$\frac{P(t + 1)}{P(t)} = \frac{5 \times 10^6 e^{0.06(t+1)}}{5 \times 10^6 e^{0.06t}} = e^{0.06} = 1.061837$$

Hence the percentage change is:

$$\frac{P(t + 1) - P(t)}{P(t)} = e^{0.06} - 1 = 0.061837,$$

so the percentage growth per annum is 6.18%.

3) Find the value of $n$ for which $(0.081)^n = 0.24$.

**Solution (5):** If $(0.081)^n = 0.24$, then taking the logarithm of each side (the base does not matter),

$$n \log(0.081) = \log(0.24),$$

or

$$n = \frac{\log(0.24)}{\log(0.081)} = 0.5678243.$$
4) Solve for $x : \log_3(x + 2) + \log_3(2x + 7) = 3$.

**Solution** (5) : If $\log_3(x + 2) + \log_3(2x + 7) = 3$, then taking each side to the power of 3,

$$3^3 = 3^{\log_3(x+2)+\log_3(2x+7)}$$

or

$$27 = (x + 2)(2x + 7)$$

This is just the quadratic equation:

$$(x - 1)(2x - 13) = 0$$

so $x = 1$ as $x = -6.5$ can not be a solution of the original equation.

5) **Spread of disease** Assume that the number of diagnosed cases of AIDS is increasing exponentially in India. The number of such cases was 88 in 2000 and 330 in 2002. Express this number in the form $ae^{bt}$, where $a$ and $b$ are constants and $t$ is time measured in years from 2000. On the basis of this model, how many cases will there be in India in the years 2015 and 2020. The population of India will be about $10^9$ by then.

**Solution** (10) : Let $AIDS(t)$ be the number of diagnosed AIDS cases in year $t+2000$. Then $AIDS(t) = ae^{bt}$. Inserting the 2000 numbers to this gives:

$$88 = AIDS(0) = ae^{0} = a$$

Inserting the 2002 numbers to this gives:

$$330 = AIDS(2) = ae^{2b}$$

or

$$b = \frac{\log(330)}{2\log(88)} = 0.6476051$$

so the complete equation is:

$$AIDS(t) = 88e^{0.6476051\times t}$$

So the number of cases in 2015 is

$$AIDS(15) = 88e^{0.6476051\times 15} = 1,456,306$$

and the number of cases in 2020 is

$$AIDS(20) = 88e^{0.6476051\times 20} = 37,111,551.$$  

For reference sake, there are about 25,000,000 cases in sub-Saharan Africa at this point.

6) **Learning curve** An individual’s efficiency in performing a routine task improves with practice. Let $t$ be the time spent learning a task and $y$ a measure of the individual’s output. (For example, $y$ might be the number of papers graded in an hour.) Then one function often used to relate $y$ to $t$ is

$$y = A(1 - e^{kt})$$

where $A$ and $k$ are constants. The graph of such a relationship is called a *learning curve*. After one hour of practice, a TA grading papers can do 10 in 50 minutes. After two hours, the person can do 15 papers in 50 minutes. Find the constants $A$ and $k$. How many papers can the person grade after
Solution (15): Inserting the one hour numbers to the learning curve gives:

\[ 10 = A(1 - e^{k \times 1}) \]

Inserting the two hour numbers to the learning curve gives:

\[ 15 = A(1 - e^{k \times 2}) \]

Dividing these two equations and re-arranging terms gives:

\[ -5 = 5e^k(2e^k - 3) \]

or

\[ (2e^k - 1)(e^k - 1) \]

and so \( k = \log_e(1/2) = -0.6931472 \). Hence \( A = 10/(1 - 2^{-1}) = 20 \). The complete equation is then

\[ y = 20(1 - e^{t \log_e(1/2)}) = 20(1 - 2^{-t}) \]

After 4 hours of practice this is:

\[ y = 20(1 - e^{4 \log_e(1/2)}) = 20(1 - 2^{-4}) = 18.75 \]