HW2: Axioms of Probability

Directions. Show and explain all work to receive full credit. Homework is due on Friday, October 14th at the beginning of class. Problem 6 is optional.

Problem 1. Given a random experiment with sample space $\Omega$. Let $E$, $F$, and $G$ be three events on $\Omega$. Find expressions for the events so that, of $E$, $F$, and $G$,

(a) both $E$ and $G$, but not $F$, occur;
(b) at least two of the events occur;
(c) at most one of the events occurs;
(d) at most two of the events occur.

Problem 2. In an experiment, two dice are thrown.

(a) Describe formally the sample space $\Omega$ of the experiment.

We consider 3 events on $\Omega$

- $E$: “the sum of the dice is odd”;
- $F$: “at least one of the dice lands on 1”;
- $G$: “the sum of the dice is 5”.

(b) Give all the outcomes of $G$.

(c) Describe in words the events $E \cap F$, $E \cup F$, $F \cap G$, $E \cap F^c$ and $E \cap F \cap G$.

Problem 3. A communication system consists of 5 antennas, each of which is either working or failed. Consider an experiment that consists of observing the status of each antenna, and let the outcome of the experiment be given by the vector $(a_1, a_2, a_3, a_4, a_5)$, where $a_i = 1$ if antenna $i$ ($i = 1, 2, 3, 4, 5$) is working and $a_i = 0$ if antenna $i$ is failed.

(a) How many outcomes are in the sample space of this experiment?

(b) Suppose that the system will work if antennas 1 and 2 are both working, or if antennas 3 and 4 are both working, or if antennas 1, 3, and 5 are all working. Let $W$ be the event that the system will work. Specify all the outcomes in $W$. 
(c) Let $F$ be the event that antennas 4 and 5 are both failed. How many outcomes are contained in $F$?

(d) Write out all the outcomes in the event $F \cap W$.

**Problem 4.** For any two events $E$ and $F$, show that

(a) if $E \subset F$, then $F^c \subset E^c$;

(b) $P(E \cup F) \geq \max(P(E), P(F))$;

(c) $P(E \cap F) \leq \min(P(E), P(F))$.

**Problem 5.** Show that the probability of that exactly one of the two events $E$ or $F$ occurs equals $P(E) + P(F) - 2P(E \cap F)$.

**Problem 6.** [Extra-Credit] Prove that for any sequence of $n$ events $(n \in \mathbb{N}, n \geq 1)$ $E_1, \ldots, E_n$,

$$P \left( \bigcup_{i=1}^{n} E_i \right) \leq \sum_{i=1}^{n} P(E_i).$$