14. INTEREST RATE DERIVATIVES: Binomial Tree Models Approach

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Reading: Tuckman (2002) Chap. 9 “Term Structure Models”
14.1 INTRODUCTION

Example

- European call option on $1000 face value of a bond with strike price $K = \text{par} = $1000$. Bond price at time $t$ is $P_t$ and bond has maturity $T$. Maturity of option is $T_0 < T$.

- Holder has the right (but not the obligation!) to buy $1000$ face value of the bond for $1000$ at option maturity, and the payoff is:

$$C = \max(P_{T_0 S} - $1000, 0)$$

Cash flow depends on value of the underlying, i.e., the level of the bond price, or equivalently the level of interest rates.
Why GBM/BSM Can Not be Used

1. The price of a bond must converge to its face value at maturity (Evolution of stock prices are not so constrained).

2. Because of the above, the volatility of a bond’s price must tend towards zero as the bond approaches maturity (Black-Scholes assumes constant volatility)

3. Interest rates are not constant. Assuming bond price is a random process and interest rates are constant makes little sense. (For stock options, it is assumed that the interest rate is constant)

NOTE: The above objections are not so serious for short-term options on long-term bonds!
These objections led researchers to make assumptions about the random evolution of interest rates, such that

\[ P_t \to \text{par} \quad \text{as } t \to T \]

\[ \sigma_t \to 0 \quad \text{as } t \to T \]

\[ r = r_t \quad \text{an appropriate random process} \]
But which interest rate??

- Ideally, one models the evolution over time of the entire term structure. But this is a very complex modeling problem.

- One main focus of research has been on “one factor” models, with the one-factor being the short rates.

- Begin with this and discuss two-factor models later
14.2 ARBITRAGE-FREE PRICING

- Allow random short rate
- Match initial term structure

6 mo. Binomial Interest Rate Tree (one-period)

The 6 mo. spot rate now is 3.99%

Also, the 12 mo. spot rate now is 4.16%.
6 Month Price Tree (one period)

\[ p = \frac{1}{2} \]

\[ 1 - p = \frac{1}{2} \]

\[ \frac{1000}{1 + \frac{0.0399}{2}} \]

\[ t = 0 \]

\[ t = \frac{1}{2} \]

\[ 980.4402 \]

\[ \$1000 \]
1 Year Price Tree (two period)

N.B. we use the 1-year spot rate and the 6 mo. forward 6 mo. rate here

\[
p = \frac{1}{2}
\]

\[
1 - p = \frac{1}{2}
\]

\[
\frac{1000}{1 + \frac{0.416}{2}}
\]

\[
\frac{1000}{1 + \frac{0.045}{2}}
\]

\[
\frac{1000}{1 + \frac{0.04}{2}}
\]

A one-year zero. Now a 6 mo. zero.
The expected value of the price at \( t = \frac{1}{2} \) is:

\[
0.5 \times (977.9951) + 0.5 \times (980.3922) = 979.1937
\]

The present value at \( t = 0 \) of the above expected value is:

\[
\frac{977.1937}{1 + \frac{0.0399}{2}} = 960.04
\]

But

\[
960.04 \neq 959.6628
\]
Why the difference?

- Investors do not price using discounted expected values

- Over the next 6 mo. the 1-yr. zero is risky. Investors would prefer 979.1937 with prob. = 1. The lower price $959.6628 reflects the penalty in price due to the risk.
Pricing a Derivative

With Previous Interest Rate and Price Trees

Call option maturing in 6 mo. to purchase $1000 face value of original 1 year zero, which is now a 6 mo. zero) at a strike price of:

\[ K = $978.50. \]

The payoff is:

\[ C = \max(P_{1/2} - 978.50, 0) \]
Price Tree for the Option

To value by arbitrage, find a replicating portfolio and value it!

\[ 0 = C \text{ (out of the money)} \]

\[ \begin{align*}
P_{\frac{1}{2},u} &= 977.9951 \\
P_{\frac{1}{2},d} &= 980.3922 \\
C &= 1.8922
\end{align*} \]

\[ t = 0 \quad \text{and} \quad t = \frac{1}{2} \]
A Replicating Portfolio

\[ F_{.5} = \text{Face value of 6-mo. zero} \]
\[ F_1 = \text{Face value of 1-yr zero} \]

What are \( F_{.5}, F_1 \)? At \( t = 1/2 \), \( F_{.5}, F_1 \) must satisfy

\[ F_{.5} + .9779951 \cdot F_1 = 0 \quad \text{Value of option in up state} \]
\[ F_{.5} + .9803922 \cdot F_1 = 1.8922 \quad \text{Value of option in down state} \]

$1$ per dollar face of the 6 mo. zero in both up and down states

$.9803922$ per dollar face value of the one-year zero in the down state
Solution:

\[ F_5 = -772.0005 \]  (short position)
\[ F_1 = 789.3705 \]  (long position)

Present value of replicating portfolio:

\[
\frac{.9804402 \cdot (-772.0005)}{\text{value of 6 mo. zeros at } t=0} + \frac{.9596628 \cdot (789.3705)}{\text{value of 1 yr zeros at } t=0} = .63
\]
Price of option is 63¢:

If not, you can make an arbitrage:

a) If price is less than 63¢ buy the option and short the replicating portfolio;

b) If price is greater than 63¢ short the option and buy the replicating portfolio.

Note once again you can’t get the price by discounting the expected future value of the option, i.e.,

\[
.5 \cdot 0 + .5 \cdot (1.8922) = .9461
\]

\[
\frac{.9461}{1 + \frac{.0399}{2}} = .9276 > .63
\]

True price is lower because of the risk!
The up and down probabilities did not enter into the calculation of the value of the option. Why?

- Because the replicating portfolio/arbitrage pricing argument is based on matching the outcomes, and prices of securities in replicating portfolio depend only on present prices of the bonds and the payoffs.
Q. So how do the probabilities enter?

A. They are “embedded” in the current bond prices:

If the probability of an up move increases, then

a) the current value of the 1 yr. zero will decrease, and

b) the value of the option would decrease.

Given the bond prices, one needn’t know the probabilities!
 Previous Example

Discounting expected price at $t = \frac{1}{2}$ does not work, at least not with $p = \frac{1}{2}$!!

14.3 RISK-NEUTRAL PRICING
Risk Neutral Pricing Using One-Year Price Tree:

Just find $p$ that satisfies

\[
\frac{977.9951 \cdot p + 980.3922 \cdot (1 - p)}{1 + \frac{.0399}{2}} = 959.6628
\]

Solution: $p = .661$

NOTE: Here the risk neutral method works just as for an option on a stock.
Risk Neutral Pricing using the 6 Month Price Tree

$$\frac{1000p + 1000(1-p)}{1 + \frac{0.0399}{2}} = \frac{1000}{1 + \frac{0.0399}{2}}$$

$p = 0.661$

$1 - p = 0.339$

$1000$

$980.4402$

$1000$

N.B. We have seen that with the risk neutral pricing approach the one and two-step price trees are consistent with the 6 mo. zero and the 1 year zero, respectively, while allowing the short rate to vary randomly, i.e., the model is consistent with the initial term structure!
The Risk-Neutral Option Price

Recall the option price tree we solved using a replicating portfolio, and replace $p = \frac{1}{2}$ with $p = 0.661$:

$$\frac{0.661 \times 0 + 0.339 \times 1.8922}{1 + \frac{0.0399}{2}} = 0.63$$

Same as before!

See also discussion on pp. 178-179 of Tuckman.
14.4 BUILDING A GENERAL TREE

Non-recombining (Two Period)

2^n nodes at step n, too complex!

- 20 steps → 1 million nodes
- 30 steps → 1 billion nodes

May be “reasonable” but computationally prohibitive!
Recombining Tree (Two Period)

$n+1$ nodes at step $n$

30 yr security with weekly steps:
$30 \times 52 + 1 = 1561$ steps
The Price Tree (Three Period)

From spot rate curve:
- 6 mo. spot = 3.99%
- 1 yr. spot = 4.16%
- 1.5 yr. spot = 4.33%

Uses 1 yr. forward 6 mo. rates

1 yr spot rate in 6 mo. is not yet known!
Determined by 1.5 yr. spot.

We have these probabilities from the 1 yr. price tree using risk neutral pricing.
Risk Neutral Pricing

\[
\frac{0.661P^u + 0.399P^d}{1 + \frac{0.0399}{2}} = 937.7641
\]

Plug second and third equations into first and solve for \( q \).

\[
P^u = \frac{978.9525q + 980.8730(1-q)}{1 + \frac{0.04}{2}}
\]

\[
P^d = \frac{976.0859q + 978.9525(1-q)}{1 + \frac{0.045}{2}}
\]

Then plug \( q \) back in to get \( P^u, P^d \).
Valuing any Security Maturing in One Year using Risk-Neutral Probabilities
On date 1, state 1, the value is
\[
\frac{0.632 \times 500 + 0.368 \times 100}{1 + 0.045/2} = 345.0367
\]

Similarly, on date 1, state 0, the value of the security is
\[
\frac{0.632 \times 100 + 0.368 \times -10}{1 + 0.04/2} = 58.3529
\]

Finally, the value of the security today is
\[
\frac{0.661 \times 345.0367 + 0.339 \times 58.3529}{1 + 0.0399/2} = 243.003
\]
Increasing the Number of Steps for Greater Accuracy

Handout Figures 8.1 – 8.4 from Tuckman (2002)
14.5 ONE-YEAR FORWARD RATES

6 mos. from now, the 1.5 yr zero becomes a 1 yr zero, with the two possible prices

- Up state: 955.6376 ⇒ 4.59% 1-yr spot
- Down state: 960.4493 ⇒ 4.08% 1-yr spot

So, the one-year spot tree is

\[
\begin{align*}
\text{Up state} & \quad 955.6376 \Rightarrow 4.59\% \quad \text{1-yr spot} \\
\text{Down state} & \quad 960.4493 \Rightarrow 4.08\% \quad \text{1-yr spot}
\end{align*}
\]

\[
\frac{F}{(1 + \frac{s(1)}{2})^2} = P
\]

\[
s(1) = 2 \left( \left( \frac{F}{P} \right)^{1/2} - 1 \right)
\]
The one-year spot rate evolution is completely determined by the assumed:

a) term structure

b) 6-mo. rate process
14.6 HO-LEE BINOMIAL TREE MODEL

- \( r \) = current short term rate
- \( \tau \) = time (in years)
- \( m \) = drift rate, changes with time (steps along tree)
- \( \sigma \) = (annual) basis point volatility

\[
\text{a random variable}
\]
Tree Building Strategy

- Want model to be arbitrage-free and give accurate derivative product prices
- $\sigma$: choose to get accurate derivative prices
  - historical data
  - a “view” on rates
  - an implied method
- $m$: choose to be arbitrage-free, i.e., model prices for zero’s match market price
For $\tau = 1$

$$E(R') = \frac{1}{2}(r + m + \sigma) + \frac{1}{2}(r + m - \sigma) = r + m$$

$$\text{Var}(R') = \frac{1}{2}(r + m + \sigma - E(R'))^2 + \frac{1}{2}(r + m - \sigma - E(R'))^2$$

$$= \frac{1}{2}\sigma^2 + \frac{1}{2}(-\sigma)^2$$

$$= \sigma^2$$

For each new step in the tree (growth):
- $m$ can change (will change)
- $\sigma$ is fixed
Suppose that: \[ \sigma = .45\% \quad (45 \text{ bp}) \]
\[ \tau = \frac{1}{2} \]
\[ \sigma \sqrt{\tau} = .0045 / \sqrt{2} \]
\[ = .00318198 \]

6 mo. rate: \( r = 3.99\% \)

1 mo. rate: 4.16%

1 year price tree

\[
p = \frac{1 + \frac{3.99\%}{2} + m + .318198\%}{1 + \frac{3.99\%}{2} + m - .318198\%}
\]
Discounted “expected” value

\[
P = \frac{0.5 \left( \frac{1}{1 + \frac{3.99\%}{2} + \frac{m}{2} + 0.318198\%} \right) + 0.5 \left( \frac{1}{1 + \frac{3.99\%}{2} + \frac{m}{2} - 0.318198\%} \right)}{1 + \frac{0.0399}{2}}
\]

“expected” value 6 mos. out: a role like that of risk-neutral probabilities

Must have

\[
P = \frac{1}{\left(1 + \frac{0.0416}{2}\right)^2} = 0.959663
\]

Why?

Solve for drift:

\[
m = 0.342089
\]
Which gives the rate tree:

\[ \frac{1}{2} \quad 4.47924\% \]

\[ \frac{1}{2} \quad 3.84285\% \]

Can use for derivative product pricing. How?
Now let’s build the tree out another step

\[ \frac{1}{2} \times 3.99\% + \frac{1}{2} \times 4.47924\% \]

\[ \frac{1}{2} \times 3.99\% + \frac{1}{2} \times 3.84285\% \]

\[ \frac{1}{2} \times 3.99\% + \frac{1}{2} \times (m + m')/2 + 2\sigma\sqrt{\tau} \]

\[ \frac{1}{2} \times 3.99\% + \frac{1}{2} \times (m + m')/2 - 2\sigma\sqrt{\tau} \]
Substitute $m = .342089$ and express the tree in the form:
The 3-step Price Tree

1.5 yr zero rate = 4.33%

\[ P = \frac{1}{\left(1 + \frac{0.0433}{2}\right)^3} = .937764 \]

\[ P_{uu} = \frac{1}{1 + \frac{0.0479744 + m/2}{2}} \]
\[ P_{ud} = \frac{1}{1 + \frac{0.0416105 + m/2}{2}} \]
\[ P_{dd} = \frac{1}{1 + \frac{0.0352465 + m/2}{2}} \]

\[ P^u = \frac{1}{2} P_{uu} + \frac{1}{2} P_{ud} \]
\[ P^d = \frac{1}{2} P_{ud} + \frac{1}{2} P_{dd} \]
And

\[ P = \frac{1}{2} P^u + \frac{1}{2} P^d \]

\[ = .937764 = \frac{1}{2} P^u + \frac{1}{2} P^d \]

\[ = \frac{1}{1 + \frac{0.0399}{2}} \]

Equation is a bit “messy”, but only \( m' \) to solve for:

\[ m' = 1.36176\% \]

**The 3-step Interest Rate Tree**

- 3.99%
  - \( \frac{1}{2} \)
    - 4.47924%
      - \( \frac{1}{2} \)
        - 5.47832%
      - \( \frac{1}{2} \)
        - 4.84193%
  - \( \frac{1}{2} \)
    - 3.84285%
      - \( \frac{1}{2} \)
        - 4.20553%