Existence of Brownian motion and Brownian bridge as continuous processes on $C[0,1]$

The aim of this subsection to convince you that both Brownian motion and Brownian bridge exist as continuous Gaussian processes on $[0,1]$, and that we can then extend the definition of Brownian motion to $[0,\infty)$.

**Definition 1.** Brownian motion (or standard Brownian motion, or a Wiener process) $S$ is a Gaussian process with continuous sample functions and:

(i) $S(0) = 0$;
(ii) $E(S(t)) = 0$, $0 \leq t \leq 1$;
(iii) $E\{S(s)S(t)\} = s \land t$, $0 \leq s, t \leq 1$.

**Definition 2.** A Brownian bridge process $U$ is a Gaussian process with continuous sample functions and:

(i) $U(0) = U(1) = 0$;
(ii) $E(U(t)) = 0$, $0 \leq t \leq 1$;
(iii) $E\{U(s)U(t)\} = s \land t - st$, $0 \leq s, t \leq 1$.

**Theorem 1.** Brownian motion $S$ and Brownian bridge $U$ exist.

**Proof.** We first construct a Brownian bridge process $U$. Let

$$h_{00}(t) \equiv h(t) \equiv \begin{cases} t, & 0 \leq t \leq 1/2, \\ 1-t, & 1/2 \leq t \leq 1, \\ 0, & \text{elsewhere}. \end{cases}$$

For $n \geq 1$ let

$$h_{nj}(t) \equiv 2^{-n/2}h(2^n t - j), \quad j = 0, \ldots, 2^n - 1.$$  

For example, $h_{10}(t) = 2^{-1/2}h(2t)$, $h_{11}(t) = 2^{-1/2}h(2t - 1)$, while

$$h_{20}(t) = 2^{-1}h(4t), \quad h_{21}(t) = 2^{-1}h(4t - 1),$$

$$h_{22}(t) = 2^{-1}h(4t - 2), \quad h_{23}(t) = 2^{-1}h(4t - 3).$$

Note that $|h_{nj}(t)| \leq 2^{-n/2}2^{-1}$.

The functions $\{h_{nj} : j = 0, \ldots, 2^n - 1, \ n \geq 0\}$ are called the Schauder functions; they are integrals of the orthonormal (with respect to Lebesgue
measure on \([0, 1]\) family of functions \(\{g_{nj} : j = 0, \ldots, 2^n - 1, \ n \geq 0\}\) called the Haar functions defined by

\[
g_{00}(t) \equiv g(t) \equiv 21_{[0,1/2]}(t) - 1,
\]

\[
g_{nj}(t) \equiv 2^{n/2}g_{00}(2^n t - j), \quad j = 0, \ldots, 2^n - 1, \ n \geq 1.
\]

Thus

\[
\int_{0}^{1} g_{nj}^2(t) dt = 1, \quad \int_{0}^{1} g_{nj}(t) g_{n'j'}(t) dt = 0 \quad \text{if} \quad n \neq n', \text{ or } j \neq j',
\]

and

\[
h_{nj}(t) = \int_{0}^{t} g_{nj}(s) ds, \quad 0 \leq t \leq 1.
\]

Furthermore, the family \(\{g_{nj}\}_{j=0}^{2^n-1} \cup \{g(\cdot /2)\}\) is complete: any \(f \in L_2(0,1)\) has an expansion in terms of the \(g\)'s. In fact the Haar basis is the simplest wavelet basis of \(L_2(0,1)\), and is the starting point for further developments in the area of wavelets.

Now let \(\{Z_{nj}\}_{j=0, n \geq 0}^{2^n-1}\) be independent identically distributed \(N(0, 1)\) random variables; if we wanted, we could construct all these random variables on the probability space \(([0, 1], \mathcal{B}_{[0,1]}, \lambda)\). Define

\[
V_n(t, \omega) = \sum_{j=0}^{2^n-1} Z_{nj}(\omega) h_{nj}(t),
\]

\[
U_m(t, \omega) = \sum_{n=0}^{m} V_n(t, \omega).
\]

For \(m > k\)

\[
|U_m(t, \omega) - U_k(t, \omega)| = |\sum_{n=k+1}^{m} V_n(t, \omega)| \leq \sum_{n=k+1}^{m} |V_n(t, \omega)|
\]

where

\[
|V_n(t, \omega)| \leq \sum_{j=0}^{2^n-1} |Z_{nj}(\omega)| |h_{nj}(t)| \leq 2^{-(n/2+1)} \max_{0 \leq j \leq 2^n-1} |Z_{nj}(\omega)|
\]

since the \(h_{nj}, j = 0, \ldots, 2^n - 1\) are \(\neq 0\) on disjoint \(t\) intervals.
Now $P(Z_{nj} > z) = 1 - \Phi(z) \leq z^{-1} \phi(z)$ for $z > 0$ (by “Mill’s ratio”) so that

$$P(|Z_{nj}| \geq 2\sqrt{n}) = 2P((Z_{nj} \geq 2\sqrt{n}) \leq \frac{2}{\sqrt{2\pi}}(2\sqrt{n})^{-1}e^{-2n}. \quad (7)$$

Hence

$$P\left(\max_{0 \leq j \leq 2^n - 1} |Z_{nj}| \geq 2\sqrt{n}\right) \leq 2^n P(|Z_{00}| \geq 2\sqrt{n}) \leq \frac{2^n}{\sqrt{2\pi}}n^{-1/2}e^{-2n}; \quad (8)$$

since this is a term of a convergent series, by the Borel-Cantelli lemma \(\max_{0 \leq j \leq 2^n - 1} |Z_{nj}| \geq 2\sqrt{n}\) occurs infinitely often with probability zero; i.e. except on a null set, for all $\omega$ there is an $N = N(\omega)$ such that $\max_{0 \leq j \leq 2^n - 1} |X_{nj}(\omega)| < 2\sqrt{n}$ for all $n > N(\omega)$. Hence

$$\sup_{0 \leq t \leq 1} |U_m(t) - U_k(t)| \leq \sum_{n=k+1}^{m} 2^{-n/2}n^{1/2} \downarrow 0 \quad (9)$$

for all $k, m \geq N' \geq N(\omega)$. Thus $U_m(t, \omega)$ converges uniformly as $m \to \infty$ with probability one to the (necessarily continuous) function

$$U(t, \omega) \equiv \sum_{n=0}^{\infty} V_n(t, \omega). \quad (10)$$

Define $U \equiv 0$ on the exceptional set. Then $U$ is continuous for all $\omega$.

Now $\{U(t) : 0 \leq t \leq 1\}$ is clearly a Gaussian process since it is the sum of Gaussian processes. We now show that $U$ is in fact a Brownian bridge: by formal calculation (it remains only to justify the interchange of summation
and expectation, 
\[
E\{U(s)U(t)\} = E\left\{ \sum_{n=0}^{\infty} V_n(s) \sum_{m=0}^{\infty} V_m(t) \right\}
\]
\[
= \sum_{n=0}^{\infty} E\{V_n(s)V_n(t)\}
\]
\[
= \sum_{n=0}^{\infty} E\left\{ \sum_{j=0}^{2^n-1} Z_{nj} \int_0^s g_{nj}d\lambda \sum_{k=0}^{2^n-1} Z_{nk} \int_0^t g_{nk}d\lambda \right\}
\]
\[
= \sum_{n=0}^{\infty} \sum_{j=0}^{2^n-1} \int_0^s g_{nj}d\lambda \int_0^t g_{nj}d\lambda
\]
\[
= \sum_{n=0}^{\infty} \sum_{j=0}^{2^n-1} \int_0^1 1_{[0,s]}(u) 1_{[0,t]}(u) du - st
\]
\[
= s \land t - st
\]
where the next to last equality follows from Parseval’s identity. Thus \(U\) is Brownian bridge.

Now let \(Z\) be one additional \(N(0,1)\) random variable independent of all the others used in the construction, and define

\[
S(t) \equiv U(t) + tZ = \sum_{n=0}^{\infty} V_n(t) + tZ. \tag{11}
\]

Then \(S\) is also Gaussian with 0 mean and

\[
Cov[S(s), S(t)] = Cov[U(s) + sZ, U(t) + tZ]
\]
\[
= Cov[U(s), U(t)] + st\text{Var}(Z)
\]
\[
= s \land t - st + st = s \land t.
\]

Thus \(S\) is Brownian motion. Since \(U\) has continuous sample paths, so does \(S\).

The following figures illustrate the construction given in the theorem.
Figure 1: The Schauder function $h_{00}$.

Here is the Mathematica code that produced the plots for the case $m = 8$.

```
Needs["Histograms'"]
ndist = NormalDistribution[0, 1]
m = 8
Z = Table[RandomReal[ndist, 2^n], {n, 0, m}]
Histogram[Flatten[Z]]

h[t_] := t /; 0 <= t <= 1/2
h[t_] := 1 - t /; 1/2 < t <= 1
h[t_] := 0 /; t < 0 || t > 1
h1[t_, n_, j_] := 2^(-n/2)*h[2^n *t - j]
V0[t_] := Z[[1, 1]]*h[t]
V[t_, n_] := Sum[Z[[n + 1, j + 1]]*h1[t, n, j], {j, 0, 2^n - 1}]
U[t_, m_] := Sum[V[t, n], {n, 1, m}] + V0[t]
P1 = Plot[h[t], {t, 0, 1}]
P2 = Plot[h1[t, 1, 0], {t, 0, 1}]
P3 = Plot[h1[t, 1, 1], {t, 0, 1}]
Show[P2, P3]
P4 = Plot[h1[t, 2, 0], {t, 0, 1}, PlotRange -> {0, .5}]
P5 = Plot[h1[t, 2, 1], {t, 0, 1}, PlotRange -> {0, .5}]
P6 = Plot[h1[t, 2, 2], {t, 0, 1}, PlotRange -> {0, .5}]
```
Figure 2: The Schauder functions $h_{10}$ and $h_{11}$.

```
P7 = Plot[h1[t, 2, 3], {t, 0, 1}, PlotRange -> {0, .5}]
Show[P4, P5, P6, P7]
Plot[V0[t], {t, 0, 1}]
Plot[V[t, 1], {t, 0, 1}, PlotStyle -> {Thickness[1/200], RGBColor[0.0, 0.0, 1.0]}]
Plot[V[t, 2], {t, 0, 1}, PlotStyle -> {Thickness[1/200], RGBColor[0.1, 0.0, 0.9]}]
Plot[V[t, 3], {t, 0, 1}, PlotStyle -> {Thickness[1/200], RGBColor[0.2, 0.0, 0.8]}]
Plot[V[t, 4], {t, 0, 1}, PlotStyle -> {Thickness[1/200], RGBColor[0.3, 0.0, 0.7]}]
Plot[V[t, 5], {t, 0, 1}, PlotStyle -> {Thickness[1/200], RGBColor[0.4, 0.0, 0.6]}]
Plot[V[t, 6], {t, 0, 1}, PlotStyle -> {Thickness[1/200], RGBColor[0.5, 0.0, 0.5]}]
Plot[V[t, 7], {t, 0, 1}, PlotStyle -> {Thickness[1/200], RGBColor[0.6, 0.0, 0.4]}]
Plot[V[t, 8], {t, 0, 1}, PlotStyle -> {Thickness[1/200], RGBColor[0.7, 0.0, 0.3]}]
Plot[U[t, 8], {t, 0, 1}, PlotStyle -> {Thickness[1/200], RGBColor[0.0, 0.5, 0.5]}]
```
Figure 3: The Schauder functions $h_{20}$, $h_{21}$, $h_{22}$, $h_{23}$.

Figure 4: A sample path of the random function $V_0(t)$. 
Figure 5: A sample path of the random function $V_1(t)$.

Figure 6: A sample path of the random function $V_2(t)$. 
Figure 7: A sample path of the random function $V_3(t)$.

Figure 8: A sample path of the random function $V_4(t)$. 
Figure 9: A sample path of the random function $V_5(t)$.

Figure 10: A sample path of the random function $V_6(t)$. 
Figure 11: A sample path of the random function $V_7(t)$.

Figure 12: A sample path of the random function $V_8(t)$.
Figure 13: A sample path of the random function $U_B(t)$.

Figure 14: A sample path of the random function $S_B(t)$. 