Handout 10: Math/Stat 394: Probability I
The Central Limit Theorem (CLT)
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Independent Repetitions: (Sampling With Replacement is a Special Case).

- The basic \( X \) experiment has mean \( \mu \) and standard deviation \( \sigma \).
- Let \( T_n \equiv X_1 + \cdots + X_n \) for independent repetitions \( X_1, \ldots, X_n \).
- \( T_n \) has mean \( n\mu \) and standard deviation \( \sqrt{n}\sigma \).
- \( X_n \) has mean \( \mu \) and standard deviation \( \sigma/\sqrt{n} \).
- The standardized random variable
  \[
  Z_n = \frac{T_n - n\mu}{\sqrt{n}\sigma} = \frac{X_n - \mu}{\sigma/\sqrt{n}}
  \]
  has mean 0 and standard deviation 1.

CLT: The distribution of \( Z_n \) converges to the \( N(0,1) \) distribution: for any real numbers \( a < b \)

\[
P(a < Z_n < b) \rightarrow P(a < Z < b)
\]

where \( Z \sim N(0,1) \).

Sampling Without Replacement: from an \( a_1, \ldots, a_N \) urn with \( \mu = \bar{a} = \sum_1^N a_i/N \) and \( \sigma = \bar{a}^2 = N^{-1} \sum_1^N (a_i - \bar{a})^2 \).

- One draw from the urn has mean \( \mu = \bar{a} \) and standard deviation \( \sigma_a \).
- Let \( T_n \equiv X_1 + \cdots + X_n \) for the dependent repetitions \( X_1, \ldots, X_n \).
- \( T_n \) has mean \( n\bar{a} \) and standard deviation \( \sqrt{n}\sigma_a \sqrt{1 - (n-1)/(N-1)} \).
- \( X_n \) has mean \( \bar{a} \) and standard deviation \( (\sigma_a/\sqrt{n}) \sqrt{1 - (n-1)/(N-1)} \).
- The standardized random variable
  \[
  Z_n = \frac{T_n - n\bar{a}}{\sqrt{n}\sigma_a \sqrt{1 - (n-1)/(N-1)}} = \frac{X_n - \bar{a}}{(\sigma_a/\sqrt{n}) \sqrt{1 - (n-1)/(N-1)}}
  \]
  has mean 0 and standard deviation 1.

Finite Sampling CLT: The distribution of \( Z_n \) converges to the \( N(0,1) \) distribution provided \( n \rightarrow \infty \) and \( N-n \rightarrow \infty \) and \( \sigma_a^2 \) does not converge to 0 as \( N \rightarrow \infty \): for any real numbers \( a < b \)

\[
P(a < Z_n < b) \rightarrow P(a < Z < b)
\]

where \( Z \sim N(0,1) \).
**Convolution formulas:** Suppose that $X$, $Y$ are independent. Let $T = X + Y$.

Then

$$ p_T(t) = \sum_{x} p_X(x) p_Y(t-x) \quad \text{or} \quad \sum_{y} p_Y(y) p_X(t-y) \quad (\text{discrete case}) $$

$$ f_T(t) = \int f_X(x) f_Y(t-x) \, dx \quad \text{or} \quad \int f_Y(y) f_X(t-y) \, dy \quad (\text{continuous case}) $$

**Proof:** Discrete case:

$$ p_T(t) = P(T = t) = \sum_{x} P(X = x, T = X + Y = t) = \sum_{x} P(X = x, Y = t - x) $$

$$ = \sum_{x} P(X = x) P(Y = t - x) \quad \text{by independence of } X, Y $$

$$ = \sum_{x} p_X(x) p_Y(t - x). $$

Continuous case: in this case we first compute the distribution of $T$. Using the independence of $X$, $Y$ it follows that the joint density of $X, Y$ is given by the product of the marginal densities: $f_{X,Y}(x, y) = f_X(x) f_Y(y)$. Then

$$ F_T(t) = P(T \leq t) = P(X + Y \leq t) = \int \int_{(x,y):x+y \leq t} f_X(x) f_Y(y) \, dx \, dy $$

$$ = \int_{x=-\infty}^{\infty} \int_{y=-\infty}^{t-x} f_X(x) f_Y(y) \, dy \, dx $$

$$ = \int_{x=-\infty}^{\infty} f_X(x) \left( \int_{y=-\infty}^{t-x} f_Y(y) \, dy \right) \, dx $$

$$ = \int_{x=-\infty}^{\infty} f_X(x) F_Y(t-x) \, dx. $$

Now by differentiating across this identity and interchanging the derivative and the integral on the right side we find that

$$ f_T(t) = F'_T(t) = \int_{-\infty}^{\infty} f_X(x) F'_Y(t-x) \, dx = \int_{-\infty}^{\infty} f_X(x) f_Y(t-x) \, dx. $$

□
Normal Distribution Facts:
A. If \( X \sim N(\mu, \sigma^2) \) and \( Y \sim N(\nu, \tau^2) \) are independent, then 
\( X + Y \sim N(\mu + \nu, \sigma^2 + \tau^2) \).
B. If \( X_1, \ldots, X_n \) are i.i.d. \( N(\mu, \sigma^2) \), then
   - \( T_n \equiv X_1 + \cdots + X_n \sim N(n\mu, n\sigma^2) \), and
   - \( \bar{X}_n \equiv \frac{1}{n}T_n \sim N(\mu, \sigma^2/n) \).

The Chi-Square Distribution:
A. If \( Z \sim N(0,1) \), then
\( X \equiv Z^2 \sim \text{Gamma}(1/2, 1/2) = \text{Chi-square}(1) \) with density
\[
f_X(x) = \frac{(x/2)^{-1/2}}{2\sqrt{\pi}} e^{-x/2} 1_{[0,\infty)}(x).
\]
B. If \( Z_1, \ldots, Z_r \) are independent \( N(0,1) \), then
\( T_r \equiv Z_1^2 + \cdots + Z_r^2 \sim \text{Gamma}(r/2, 1/2) = \text{Chi-square}(r) \) with density
\[
f_{T_r}(x) = \frac{(x/2)^{r/2-1}}{2\Gamma(r/2)} e^{-x/2} 1_{[0,\infty)}(x).
\]
C. If \( X = Z^2 \) with \( Z \sim N(0,1) \) (so that \( X \sim \text{Chi-square}(1) \)), then \( E(X) = E(Z^2) = 1 \) and \( \text{Var}(X) = E(Z^4) - (E(Z^2))^2 = 3 - 1 = 2 \). Thus for \( T_r \sim \text{Chi-square}(r) \), \( E(T_r) = r \) and \( \text{Var}(T_r) = 2r \).

Proof of A and B:
\[
F_X(x) = P(X \leq x) = P(Z^2 \leq x) = P(-\sqrt{x} \leq Z \leq \sqrt{x}) = 2(P(Z \leq \sqrt{x}) - P(Z \leq 0)) = 2\Phi(\sqrt{x}) - 1.
\]
Thus for \( x > 0 \),
\[
f_X(x) = 2\phi(\sqrt{x})(1/2)x^{-1/2} = \frac{x^{-1/2}}{\sqrt{2\pi}} e^{-x/2}.
\]
This is the Gamma(1/2, 1/2) density, and this gets the special name Chi-square(1). B follows from A and the duplication property of the Gamma distribution. The only thing new in C is \( E(Z^4) = 3 \). But we have
\[
E(Z^4) = \int_{-\infty}^{\infty} z^4 \phi(z) dz = 2 \int_{0}^{\infty} \frac{z^4}{\sqrt{2\pi}} e^{-z^2/2} dz
\]
\[
= 2 \int_{0}^{\infty} \frac{(2t)^{3/2}}{\sqrt{2\pi}} e^{-t} dt
\]
Figure 1: Plot of Chi-square\((k, 1)\) densities \(k = 1, 3, 6, 8, 10, 12\).

\[
\begin{align*}
\Gamma(5/2) &= (3/2)\Gamma(3/2) = (3/2)(1/2)\Gamma(1/2) = (3/2)(1/2)\sqrt{\pi}; \\
\text{see Kelly pages 490-491.}
\end{align*}
\]

\(\square\)
Figure 2: Plot of Standardized Chi-square($k,1$) densities $k = 1, 4, 8, 16, 32, 64$. 