Statistics 492, Problem Set 1
Wellner; 1/07/14

**Reading:** Karlin and Taylor, chapter 7, pages 340-365
Karlin and Taylor, chapter 6, section 8, pages 313-325.

**Due:** Tuesday, 14 January 2014.

1. Let \( \{ B(t) : t \geq 0 \} \) be Brownian motion started at \( x \). We showed in class that
\[
P_x(B(t_1) \leq x_1, B(t_2) \leq x_2) = \int_{-\infty}^{x_1} \left( \int_{-\infty}^{x_2} p_{t_2-t_1}(y_1, y_2) dy_2 \right) p_{t_1}(x, y_1) dy_1
\]
where \( p_t(x, y) \) is the transition probability density for Brownian motion. Generalize this formula to \( m \geq 3 \) time points with \( 0 \leq t_1 \leq \cdots \leq t_m \) and levels \( x_1, \ldots, x_m \).

2. Let \( B \) denote standard Brownian motion starting from 0 at time 0, and let \( \alpha \in \mathbb{R} \). Show that \( Y_\alpha(t) \equiv \exp(-\alpha t)B(e^{2\alpha t}) \) is a Gaussian process and compute its covariance function.

3. Let \( B \) denote standard Brownian motion. Show that \( \{ U(t) = B(t) - tB(1) : 0 \leq t \leq 1 \} \) is a Gaussian process. Find the covariance function of the process \( U \).

4. Let \( B \) denote Brownian motion, and define a new process \( V \) by \( V(t) = (1 - t)B(t/(1-t)) \) for \( 0 \leq t \leq 1 \). Show that \( V \) is a Gaussian process and compute the covariance function of the process \( V \). (The process \( V \) is a Brownian bridge process on \([0,1]\).)

5. Let \( \{ U(t) : 0 \leq t \leq 1 \} \) be a Brownian bridge process on \([0,1]\), and consider the process \( W(t) = (1+t)U(\frac{t}{1+t}) \) for \( 0 \leq t < \infty \). Show that \( W \) is a Brownian motion process on \([0,\infty)\). (This is called Doob’s transformation.)

6. Let \( B(t) \) denote standard Brownian motion, let \( t_j = j/m \) for \( j = 0, \ldots, m \), and consider the Riemann sums \( Y_m = \sum_{j=0}^{m} B(t_j)(t_{j+1} - t_j) \) approximating \( \int_0^1 B(t)dt \). Compute \( Var(Y_m) \equiv \sigma_m^2 \) and show that \( \lim_{m \to \infty} \sigma_m^2 = 1/3 \).

7. **Optional bonus problem:** Karlin and Taylor, page 384, problem 9: If \( B \) denotes standard Brownian motion, derive the conditional distribution of \( W = \int_0^t B(s)ds \) given that \( B(t) = x \).