1. Show that the transition density for Brownian motion,
   \[ p_t(x, y) = (2\pi t)^{-1/2} \exp\left(-\frac{(y-x)^2}{2t}\right), \]
   satisfies the "heat equation"
   \[ \frac{\partial p}{\partial t} = \frac{1}{2} \frac{\partial^2 p}{\partial t^2}. \]

2. Klebaner, Exercise 3.12, page 86: Formulate the law of large numbers and the law of the iterated logarithm for Brownian motion near zero. [Hint: Use the fact that if \( B \) is standard Brownian motion, then \( W(t) = tB(1/t), t > 0, \) and \( W(0) = 0 \) is also Brownian motion.]

3. (i) Let \( f(x, t) \equiv x^4 - 6x^2t + 3t^2 \) and let \( B \) denote standard Brownian motion. Show that \( \{f(B(t), t) : t \geq 0\} \) is a martingale. [Hint: Use the exponential martingale \( Y_c(t) \equiv \exp(cB(t) - ct^2/2) \) and compute \( (\partial^4/\partial c^4)(Y_c)|_{c=0} \).]

   (ii) Show that \( f(x, t) \) given in (i) satisfies the "backwards heat equation"
   \[ \frac{\partial f}{\partial t} = -\frac{1}{2} \frac{\partial^2 f}{\partial t^2}. \]

   It turns out that for any polynomial function \( f \) in \( x \) and \( t \) which satisfies the backwards heat equation, \( f(B(t), t) \) is a martingale.

4. Derive the joint distribution of \( B(t) \) and \( m(t) \equiv \min_{0 \leq s \leq t} B(s) \). [Hint: Consider the process \(-B\) and the joint distribution of \( (B(t), M(t)) \) where \( M(t) \equiv \max_{0 \leq s \leq t} B(s) \).]

5. Optional bonus problem: Let \( T_a \equiv \inf\{t > 0 : B_t \notin (-a,a)\} \). Show that \( E(T_a) = a^2 \) and that \( E(T_a^2) = 5a^4/3 \). Conclude that \( \text{Var}(T_a) = 2a^4/3 \). [Hint: Use the martingale in problem 3 above.]