Reading:  Klebaner, chapter 4, pages 102-116, 
Klebaner, chapter 7, pages 169-177.

Reminder:  No Problem Set 5 next week; 
Problems Set 5 will be handed out on Tuesday 18 February.

Lectures by Shuliu Yuan on February 4, 6, 11 will cover renewal theory 
··· based on Durrett, chapter 3; Karlin and Taylor, chapter 5

Please work on your projects during the next two weeks.

Due:  Tuesday, 4 February 2014.

1. Klebaner, Exercise 4.1, page 114: Let \( \tau_1 < \tau_2 \) be stopping times with respect to the natural filtration \( \{ \mathcal{F}_t \}_{t \geq 0} \) of Brownian motion \( B \) on \([0, T]\). Show that \( X(t) = 1_{(\tau_1, \tau_2]}(t) \) is a simple predictable process.

2. Klebaner, Exercise 4.5, page 115: Show that if \( X(t, s) \) is a non-random function of both \( s \) and \( t \) with \( \int_0^t X^2(t, s)ds < \infty \), then \( Y(t) = \int_0^t X(t, s)dB(s) \) is a Gaussian random variable \( Y(t) \), and the process \( \{ Y(t) : 0 \leq t \leq T \} \) is a Gaussian process with zero mean and covariance function given by \( Cov(Y(t), Y(t + v)) = \int_0^t X(t, s)X(t + v, s)ds \) for \( v \geq 0 \).

3. Let \( B \) be standard Brownian motion and let \( x \in \mathbb{R} \). Define a new process \( Y_t \) by
\[
Y_t = e^{-t/2}x + e^{-t/2} \int_0^t e^{s/2}dB_s.
\]

(i) Show that \( \{ Y_t : t \geq 0 \} \) is a Gaussian process with mean \( e^{-t/2}x \) and variance \( 1 - e^{-t} \).

(ii) Now let \( Z \sim N(0, 1) \) be independent of \( B \) and define \( \tilde{Y}(t) \) by
\[
\tilde{Y}_t = e^{-t/2}Z + e^{-t/2} \int_0^t e^{s/2}dB_s.
\]

Show that \( \tilde{Y} \) is a mean 0 Gaussian process with variance 1, and show that \( Y(t) \overset{d}{=} e^{-t/2}x + e^{-t/2}B(e^t - 1) \). Thus with \( Z(t) \equiv \int_0^t e^{s/2}dB(s) \), \( Z(t) \overset{d}{=} B(e^t - 1) \) as processes, or, equivalently, \( Z(\log(t + 1)) \overset{d}{=} B(t) \).

(iii) Compute the covariance of the process \( \tilde{Y} \). The processes \( Y \) and \( \tilde{Y} \) are known as Ornstein-Uhlenbeck processes.
4. Let \( X_n \) be a sequence of random variables with normal distributions \( N(\mu_n, \sigma_n^2) \) and suppose that \( X_n \to_d X \). (i) Show that the distribution of \( X \) is either normal or degenerate (i.e. \( P(X = x_0) = 1 \) for some \( x_0 \in \mathbb{R} \)).
(ii) Show that if \( E(X_n) \to \mu \) and \( \text{Var}(X_n) \to \sigma^2 > 0 \), then the limiting distribution (of \( X \)) is \( N(\mu, \sigma^2) \).
(iii) Since convergence in probability implies convergence in distribution, deduce convergence of Itô integrals of simple deterministic processes to a Gaussian limit.

5. Use the Itô isometry to calculate the variances of
\[
\int_0^t |B_s|^{1/2} dB_s \quad \text{and} \quad \int_0^t (B_s + s)^2 ds.
\]

6. **Optional bonus problem:** The integrals
\[
I_1 = \int_0^t B(s) ds \quad \text{and} \quad I_2 = \int_0^t B(s)^2 ds
\]
are not stochastic integrals, although they are random variables (and define natural stochastic processes). For each \( \omega \) the integrands are nice continuous functions of \( s \) and the \( ds \) integration is just the traditional calculus integration. Find the mean and variance of the the random variables \( I_1 \) and \( I_2 \).