Cube-root asymptotics for Hammersley’s process

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Hammersley’s process with sources and sinks:

Sources are Poisson with intensity $\lambda$.
Sinks are Poisson with intensity $1/\lambda$.
$\alpha$-points ($\times$) are Poisson with intensity 1.
Consider symmetric case ($\lambda = 1$).

Longest weakly NE paths to ($t, t$)

Note that here $Z'(t) < 0$. Furthermore,

$$Z(t) \overset{\mathcal{D}}{=} -Z'(t).$$
Define $L(x, t)$ as the length of a longest weakly NE path to $(x, t)$ in the symmetric case.

**Theorem:**

$$\text{Var}(L(x, t)) = -x + t + 2\mathbb{E}Z_+.$$ 

**Proof:** Introduce 4 directions $S, W, N$ and $E$.

$$L(x, t) = S + E = N + W.$$ 

Burke’s Theorem states that $N$ and $E$ are independent, so

$$\text{Var}(L(x, t)) = \text{Var}(W) - \text{Var}(N) + 2\text{Cov}(N, S) = -x + t + 2\text{Cov}(N, S).$$

Take a source-intensity $1 + \varepsilon$. Condition on $S$:

$$\text{Cov}(N, S) = \frac{\partial}{\partial \varepsilon} \bigg|_{\varepsilon=0} \mathbb{E}_\varepsilon(N) = \frac{\partial}{\partial \varepsilon} \bigg|_{\varepsilon=0} \mathbb{E}_\varepsilon(N+W).$$

This derivative corresponds to adding an extra source uniformly on $[0, x]$. An extra source at $y$ only leads to an increase of $L(x, t)$ if $y \leq Z_+$. 
Rescaling and monotonicity gives for $\lambda \geq 1$

**Corollary**

$$\text{Var}(L_\lambda(t,t)) \leq (\lambda - 1/\lambda)t + \text{Var}(L(t,t)).$$

Important because of following idea. Define

$$N(z) = \text{number of sources in } [0, z] \times \{0\}$$

and

$$A_t(z) = \text{length of longest strictly NE path from } (z, 0) \text{ to } (t, t).$$

Then $N$ and $A_t$ are independent processes and

$$L(t,t) = \sup\{N(z) + A_t(z) : -t \leq z \leq t\}.$$

Note that $N$ is just a Poisson process. Furthermore, we can control the process $A_t$. 

$L_\lambda$ has thickened set of sources ($\lambda > 1$) and thinned set of sinks ($1/\lambda$).

**Lemma:**

$$A_t(z) \leq L_\lambda(t,t) - L_\lambda(z,0).$$

$$\mathbb{E}A_t(z) \leq 2\sqrt{t(t - z)} \approx 2t - z - z^2/4t.$$
\[ P \{ Z(t) > u \} \]
\[ = P \{ \exists z > u : N(z) + A_t(z) = L(t, t) \} \]
\[ \leq P \{ \exists z > u : N(z) + L_\lambda(t, t) - L_\lambda(z, 0) \geq L(t, t) \} \]
\[ = P \{ \exists z > u : L_\lambda(z, 0) - N(z) \leq L_\lambda(t, t) - L(t, t) \}. \]

Note that \( \tilde{N}(z) = L_\lambda(z, 0) - N(z) \) is a Poisson process with intensity \( \lambda - 1 \). Therefore

\[ P \{ Z(t) > u \} \leq P \{ \tilde{N}(u) \leq L_\lambda(t, t) - L(t, t) \}. \]

Optimize \( \lambda \):

\[ \lambda = (1 - u/t)^{-1/2}. \]
\[ E\tilde{N}(u) - E\{ L_\lambda(t, t) - L(t, t) \} \geq \frac{1}{4}u^2/t. \]
\[ \text{Var}\tilde{N}(u) \leq u. \]
\[ \text{Var}(L_\lambda(t, t) - L(t, t)) \leq 8EZ(t)_+ + 4u. \]
Chebyshev:
\[ \mathbb{P}\{Z(t) > u\} \lesssim \frac{t^2}{u^3} + \frac{t^2 \mathbb{E} Z(t)_+}{u^4}. \]
Take \( u = c \mathbb{E} Z(t)_+ \). Then
\[ \mathbb{P}\{Z(t) > c \mathbb{E} Z(t)_+\} \lesssim \frac{t^2}{\mathbb{E} Z(t)_+^3} \left( \frac{1}{c^3} + \frac{1}{c^4} \right). \]

If
\[ \frac{\mathbb{E} Z(t_n)_+^3}{t_n^2} \to \infty, \]
then Dominated Convergence shows that
\[ \mathbb{E} \left( \frac{Z(t_n)_+}{\mathbb{E} Z(t_n)_+} \right) \to 0, \]
which is absurd. Therefore,
\[ \limsup_{t \to \infty} \frac{\mathbb{E} Z(t)_+}{t^{2/3}} < +\infty. \]
Consider the original Hammersley process. \( L_0(x, t) \) is length of longest path to \((x, t)\).

\[
A_t(0) - A_t(z) \overset{D}{=} L_0(t, t) - L_0(t - z, t).
\]
Corollary:

\[ \lim \inf_{t \to \infty} \frac{\mathbb{E}Z(t) + t^2/3}{t^{2/3}} > 0. \]

Suppose \( t_n \to \infty \) such that
\[ \frac{\mathbb{E}Z(t_n) + t_n^2/3}{t_n^{2/3}} \to 0. \]

Then
\[ \mathbb{P}\{Z(t_n) > \varepsilon t_n^{2/3}\} \leq \frac{\mathbb{E}Z(t_n) + \varepsilon t_n^{2/3}}{\varepsilon t_n^{2/3}} \to 0. \]

Since \(-Z'(t) \overset{D}{=} Z(t)\) and \( Z'(t) \leq Z(t) \), we have that \( \mathbb{P}\{Z(t) \geq 0\} \geq 1/2 \). This would mean that for all \( \varepsilon > 0 \),
\[ \lim \inf_{n \to \infty} \mathbb{P}\{0 \leq Z(t_n) \leq \varepsilon t_n^{2/3}\} \geq \frac{1}{2}, \]
which contradicts the previous Theorem.