Empirical Processes Working Group  
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Three Problems  
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1 Problem 1

Example 1. ($L_p$ deviations about the sample mean). Let $X, X_1, X_2, \ldots, X_n$ be i.i.d. $P$ on $\mathbb{R}$ and let $P_n$ denote the empirical measure of the $X_i$’s:

Let $\bar{X}_n = n^{-1} \sum_{i=1}^{n} X_i$, and, for $p \geq 1$ consider the $L_p$ deviations about $\bar{X}_n$:

$$A_n(p) = \frac{1}{n} \sum_{i=1}^{n} |X_i - \bar{X}|^p = P_n |X - \bar{X}_n|^p.$$ 

Questions:
(i) Does $A_n(p) \rightarrow_p E|X - E(X)|^p \equiv a(p)$?
(ii) Does $\sqrt{n}(A_n(p) - a(p)) \rightarrow_d N(0, V^2(p))$? And what is $V^2(p)$?

As will become clear, to answer question (i) we will proceed by showing that the class of functions $G_\delta \equiv \{x \mapsto |x - t|^p : |t - \mu| \leq \delta\}$ is a $P$–Glivenko-Cantelli class, and to answer question (ii) we will show that $G_\delta$ is a $P$–Donsker class.

Example 1p. ($L_p$–deviations about the sample mean considered as a process in $p$). Suppose we want to study $A_n(p)$ as a stochastic process indexed by $p \in [a, b]$ for some $0 < a \leq 1 \leq b < \infty$. Can we prove that

$$\sup_{a \leq p \leq b} |A_n(p) - a(p)| \rightarrow_{a.s.} 0?$$

Can we prove that

$$\sqrt{n}(A_n - a) \Rightarrow \mathcal{A} \quad \text{in} \quad D[a, b]$$

as a process in $p \in [a, b]$? This will require study of the empirical measure $P_n$ and empirical process $\mathcal{G}_n$ indexed by the class of functions

$$\mathcal{F}_\delta = \{f_{t,p} : |t - \mu| \leq \delta, a \leq p \leq b\}$$

where $f_{t,p}(x) = |x - t|^p$ for $x \in \mathbb{R}, t \in \mathbb{R}, p > 0$.

Example 1d. ($p$–th power of $L_q$ deviations about the sample mean). Let $X, X_1, X_2, \ldots, X_n$ be i.i.d. $P$ on $\mathbb{R}^d$ and let $P_n$ denote the empirical measure of the $X_i$’s:
Let $\overline{X}_n = n^{-1} \sum_{i=1}^n X_i$, and, for $p, q \geq 1$ consider the deviations about $\overline{X}_n$ measured in the $L_q$-metric on $\mathbb{R}^d$:

$$A_n(p, q) = \frac{1}{n} \sum_{i=1}^n \|X_i - \overline{X}_n\|_q = \mathbb{P}_n \|X - \overline{X}_n\|_q^p$$

where

$$\|x\|_q = (|x_1|^q + \cdots + |x_d|^q)^{1/q}.$$ 

Questions:
(i) Does $A_n(p) \to_p E\|X - E(X)\|_q^p = a(p, q)$?
(ii) Does $\sqrt{n}(A_n(p, q) - a(p, q)) \to_d N(0, V^2(p, q))$? And what is $V^2(p, q)$?

2 Problem 2.

Example 2. Least $L_p$-estimates of location. Now suppose that we want to consider the measure of location corresponding to minimum $L_p$-deviation:

$$\hat{\mu}_n(p) \equiv \arg\min_t \mathbb{P}_n \|X - t\|^p$$

for $1 \leq p < \infty$. Of course $\hat{\mu}_n(2) = \overline{X}_n$ while $\hat{\mu}_n(1)$ is any median of $X_1, \ldots, X_n$. The asymptotic behavior of $\hat{\mu}_n(p)$ is well-known for $p = 1$ or $p = 2$, but for $p \neq 1, 2$ it is perhaps not so well-known. Consistency and asymptotic normality for any fixed $p$ can be treated as a special case of the argmax (or argmin) continuous mapping theorem which we will introduce as an important tool in chapter/lecture 2. The analysis in this case will again depend on various (Glivenko-Cantelli, Donsker) properties of the class of functions $\mathcal{F} = \{f_t(x) : t \in \mathbb{R}\}$ with $f_t(x) = |x - t|^p$.

Example 2p. Least $L_p$-estimates of location as a process in $p$. What can be said about the estimators $\hat{\mu}_n(p)$ considered as a process in $p$, say for $1 \leq p \leq b$ for some finite $b$? (Probably $b = 2$ would usually give the range of interest.)

Example 2d. Least $p$-th power of $L_q$-deviation estimates of location in $\mathbb{R}^d$. Now suppose that $X_1, \ldots, X_n$ are i.i.d. $P$ in $\mathbb{R}^d$. Suppose that we want to consider the measure of location corresponding to minimum $L_q$-deviation raised to the $p$-th power:

$$\hat{\mu}_n(p, q) \equiv \arg\min_t \mathbb{P}_n \|X - t\|_q^p$$

for $1 \leq p, q < \infty$. 

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3 Problem 3.

Example 9.B. (Kendall’s process). Suppose that $X \sim P$ on $R^2$ with distribution function $H$ and marginal distributions $F_1$ and $F_2$. Then there is always a distribution function $C$ on $[0,1]^2$ with uniform marginal distributions (a copula function) such that

$$H(x_1, x_2) = C(F_1(x_1), F_2(x_2)).$$

Suppose that $X, X_1, \ldots, X_n$ are i.i.d. with distribution function $H$, let $\epsilon = H(X)$, $\epsilon_i = H(X_i)$, and let $Q$ denote the distribution function of the $\epsilon$’s:

$$Q(t) = P(H(X) \leq t), \quad 0 \leq t \leq 1.$$

A natural estimator of $H$ is the empirical distribution function $H_n$ of the $X_i$’s, and hence the pseudo observations are

$$\hat{\epsilon}_{n,i} = H_n(X_i) = \frac{1}{n} \# \{j \leq n : X_j \leq X_i \},$$

and the empirical distribution function of the $\hat{\epsilon}_{n,i}$’s is

$$\hat{Q}_n(t) = \frac{1}{n} \sum_{i=1}^{n} 1_{[0,t]}(\hat{\epsilon}_{n,i}),$$

the empirical measure of the $\hat{\epsilon}_{n,i}$’s indexed by the class of indicator functions $G = \{1_{[0,t]} : t \in [0,1]\}$. In this case it is easily seen that $Q$ and $\hat{Q}_n$ do not depend on the marginal distributions $F_1, F_2$ of $H$, but only on the copula function $C$. This example has been considered in detail in Barbe, Genest, Ghoudi, and Rémillard (1996) and Ghoudi and Rémillard (1998). Questions:

(i) Does $Q_n(t) \rightarrow_p Q(t)$ uniformly in $t \in [0,1]$?
(ii) Does $\sqrt{n}(Q_n(t) - Q(t)) \Rightarrow Q(t)$ for some Gaussian process $Q$? [Yes! See Barbe, Genest, Ghoudi, and Rémillard (1996). But what is going on? Can the proof be simplified? What is the relationship to the class of “lower-layers”?]

References


