1 Existence of UMPs in light of a changing significance level

This problem comes tweaked from *Testing Statistical Hypotheses 3rd ed.* by Lehmann and Romano.

Let $P_0$, $P_1$, and $P_2$ be probability mass functions assigning to the integers 1, \ldots, 6 the following probabilities:

<table>
<thead>
<tr>
<th>$X$</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>$P_0(X=x)$</td>
<td>0.02</td>
<td>0.02</td>
<td>0.02</td>
<td>0.01</td>
<td>0.00</td>
<td>0.93</td>
</tr>
<tr>
<td>$P_1(X=x)$</td>
<td>0.05</td>
<td>0.06</td>
<td>0.08</td>
<td>0.02</td>
<td>0.01</td>
<td>0.78</td>
</tr>
<tr>
<td>$P_2(X=x)$</td>
<td>0.08</td>
<td>0.05</td>
<td>0.06</td>
<td>0.00</td>
<td>0.02</td>
<td>0.79</td>
</tr>
</tbody>
</table>

We’re going to work to determine whether or not there exists uniformly most powerful level-\(\alpha\) tests of the form

$\text{Test}_0 \rightarrow H_0 : P = P_0 \text{ vs } H_1 : P \neq P_0$

for (i) $\alpha = 0$, (ii) $\alpha = 0.04$, and (iv) $\alpha = 0.06$.

(a) *Write down what you think it means to have a uniformly most powerful test. Check if your definition agrees with others in your group.*

A uniformly most powerful test is a hypothesis test which has the greatest power among all possible tests of a given size $\alpha$. For composite tests, it must be simultaneously most powerful over all simple/point alternative hypotheses that compose the composite alternative hypothesis.
(b) Draw two plots corresponding to the likelihood ratios that you would use if you were performing the point alternative tests
\[ \text{Test}_1 \rightarrow H_0 : P = P_0 \ \text{vs} \ \text{H}_1 : P = P_1 \ \text{and} \ \text{Test}_2 \rightarrow H_0 : P = P_0 \ \text{vs} \ \text{H}_1 : P = P_2. \]

(c) Which values of the two likelihood ratio plots that you just drew will result in rejection of the null hypothesis for any reasonable test? What values of \( X \) results in these values? Does it make sense in these two tests that we would reject the null if we observed these values of \( X \)?

We’d reject for \( LR = \infty \). This corresponds to \( X=5 \). It makes sense because \( P_0(X = 5) = 0 \) but \( P_1(X = 5) > 0 \) and \( P_2(X = 5) > 0 \). So, if we observe a 5, we can be sure that the true data-generating distribution is not \( P_0 \).

(d) Consider \( \alpha = 0 \). Define the critical regions (in terms of values of \( X \)) for \( \text{Test}_1 \) and \( \text{Test}_2 \). Call the regions \( C_1 \) and \( C_2 \) respectively.

Since we know from the N-P Lemma that our most powerful tests will be of the form: “Reject \( H_0 \) if \( LR_{1/0} = L_{\text{alt}}(X = x)/L_{\text{null}}(X = x) > K \)”, we must add points to our critical region in order of the size of the likelihoods. So, we first add \( X = 5 \) to both critical regions. We can calculate the probability of type I error \( (P\text{type I error}) = P_0(X \in C) = P_0(X = 5) = 0 \), and it is still zero, so we can keep 5 in the critical regions. This is equivalent to setting \( K = 10 \), or any number larger than 4.

Next, for \( \text{Test}_1 \), we would have to add \( X = 3 \) next, and for \( \text{Test}_2 \) we would add \( X = 1 \). But, in both cases the probability of type I error will now be 0.02, so we can’t keep either of these values in the critical regions because we’ve specified that the probability of type I error must be less than or equal to \( \alpha = 0 \). So,

\[ C_{1,\alpha=0} = \{5\} \]
\[ C_{2,\alpha=0} = \{5\} \]
(e) **Does a level-$\alpha = 0$ test UMP test exist for $T_{0}$? If yes, what is the test and what power does it have? If no, why not?**

We do have a UMP test in this situation. It says reject if $X = 5$ (or if $LR = \infty$).

We know that $T_{1}$ and $T_{2}$ are both most powerful from the N-P lemma, and since they’re the same test, regardless of which alternative we use, we have a UMP test for $T_{0}$.

The power needs to be calculated under a specific alternative. $Power_1 = P_{1}(\text{reject}H_0) = P_{1}(X = 5) = 0.01$. Similarly $Power_2 = 0.02$.

(f) **Repeat parts (d) and (e) for $\alpha = 0.04$**

\[
C_{1,0.04} = \{5, 3, 2\} \\
C_{2,0.04} = \{5, 1, 3\}
\]

Now we don’t have a UMP! If we wanted to use a most powerful test, it would depend on which alternative we wanted to test.

(g) **Repeat parts (d) and (e) for $\alpha = 0.06$**

\[
C_{1,0.04} = \{5, 3, 2, 1\} \\
C_{2,0.04} = \{5, 1, 3, 2\}
\]

We again have a most powerful test!! We reject if our observation is a 1, 2, 3, or 5. Now $Power_1 = 0.2$, and $Power_2 = 0.21$. 