Completeness and Basu’s theorem

Definition: Let $T$ be a statistic. It is complete if for all bounded functions $g$ we have $E_g(T) = 0 \iff g = 0$ a.s.

Note: Completeness is a property of the distribution of $T$, rather than of the statistic itself.

Example: Let $X$ follow a natural exponential family with $\text{int}(\mathcal{F})$ non-empty. Recall that the density of the natural sufficient statistic $T(X)$ is of the form $f_T(t;\theta) = a(\theta)b(t)\exp(\theta t)$

Assume that $\int g(t)a(\theta)b(t)\exp(\theta t)dt = 0$. Define $d\theta(t) = g^*(t)b(t)dt$ where $g^*(t) = \max\{\pm g(t), 0\}$. Then $\int \exp(\theta t)d\theta(t) = \int \exp(\theta t)d\theta^*(t)$, so by the uniqueness theorem for Laplace transforms $\theta^* = \theta$, whence $g(t) = -g(t)$ so $g(t) = 0$.

Definition: A statistic $U$ is conditionally ancillary given a statistic $V$ if $\mathcal{L}(U|V)$ is free of $\theta$.

Example: If $U = X$ then $V$ is sufficient, while if $V$ is constant then $U$ is ancillary.

Theorem 1.2: If $V$ is complete, there are no non-trivial conditionally ancillary statistics given $V$.

Proof: Let $A$ be a set defined in terms of $U$. Compute

$\int P(A|V = v)\mathbb{1}_A(v)f_U(v;\theta)dv = E_{\theta}(P(A|V = v))P_U(A) = P_U(A)P_U(A) = 0$

By completeness $P(A|V) = \mathbb{1}_A$ a.s., so $U$ must be a constant a.s.

Theorem 1.3: Let $T$, $U$, $V$ be statistics. Suppose that $(T, V)$ is sufficient, and that $U$ is conditionally ancillary, given $V$. If the distribution of $(T, V)$ is complete, then $T$ and $U$ are conditionally independent, given $V$.

Corollary (Basu’s theorem): Suppose $T$ is sufficient, and $U$ is ancillary. Then if $T$ is complete, $T$ and $U$ are independent.

Proof: $\int P(U \square u | T = t, V = v)P(U \square u | V = v)f_{T,V}(t, v; \theta)dtdv$

$= E_{\theta}P(U \square u | T, V)E_{\theta}P(U \square u | V) = P_{U}(U \square u)P_{U}(U \square u) = 0$

so by the completeness the conditional distribution of $U$ given $(T, V)$ is the same as the conditional distribution of $U$ given $V$. 