Homework 3
Due January 28

1. (a) Let \( f_{X,Y}(x,y;q) = \exp(-qx + y/q), x, y > 0 \). Based on a sample of size \( n \) find a sufficient statistic for \( q \), the mle for \( q \), and the Fisher information about \( q \) in the sample.

(b) Find the density for the mle, and determine how much information is lost by observing only the mle (rather than the entire sample).

(c) Let \( U = \sqrt[4]{X_i Y_i} \). Show that \( U \) is ancillary for \( q \). For extra credit, determine the density of \( U \).

2. Let \( X_1, \ldots, X_n \) be iid exponential random variables with mean \( q \).
   
   (a) Assume that the sum of the observations is 20. Using \( \frac{\hat{q} - q}{\sqrt{\hat{J}_4}} \) as an approximately standard normal pivot, derive an approximate confidence interval for \( q \).
   
   (b) Let \( \hat{q} = x^{1/3} \). Construct an approximate confidence interval for \( q \) by using the asymptotic pivot \( \frac{\hat{q} - q}{\sqrt{\hat{J}_4(q)}} \) where \( \hat{J}_4 \) is the observed information in the new parametrization.
   
   (c) Find exact upper and lower limits for a confidence interval for \( q \), and compare to the approximations in (a) and (b).

3. Consider two experiments, \( E_1 \) and \( E_2 \), consisting of observing \( X_1 \) and \( X_2 \), respectively, each with possible values \( \{1,2,3\} \) and parameter values \( \{0,1\} \). The densities are

   \[
   \begin{align*}
   X_1: & \quad \begin{array}{ccc}
   & 1 & 2 & 3 \\
   f_0^1(x_1) & .90 & .05 & .05 \\
   f_1^1(x_1) & .09 & .055 & .855 \\
   \end{array} \\
   \end{align*}
   \]

   \[
   \begin{align*}
   X_2: & \quad \begin{array}{ccc}
   & 1 & 2 & 3 \\
   f_0^2(x_2) & .26 & .73 & .01 \\
   f_1^2(x_2) & .026 & .803 & .171 \\
   \end{array} \\
   \end{align*}
   \]

   (a) Show that the test accepting \( q = 0 \) when \( x = 1 \) is most powerful in both experiments, and compute the type I and type II error probabilities for this test.
   
   (b) Show that the likelihood principle deems the experiments indistinguishable.
   
   (c) Which experiment would you rather perform?