REDUCED RANK SPATIAL COVARIANCE: A MULTiresolution APPROACH

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Summary

1 Reduced Rank Covariance
   - Wavelet multiresolution approach
   - The reduced rank covariance
   - Applications: artificial data, AOT and ozone data
   - Conclusions

2 References
The problem of large data sets...

... we propose a wavelet based method for reducing rank of covariance matrices of non stationary and incomplete data sets.

Examples of large data sets:

- data spatially distributed on a regular grid with many missing data (satellite data: aerosols, radar, ndvi, ecc.);
- irregularly distributed observations (gauge data: ozone, rainfall, etc.)
The general concept behind this work is the expansion of a random function in terms of basis functions

\[ f = W \gamma \]

where \( W \) is a matrix of basis and \( \gamma \) is the vector of coefficients.

- **Fourier basis representation** (Panciorek, 2006)
  \( W \) is the matrix of orthogonal spectral basis functions, and \( \gamma_k = a_k + b_k, \ k = 1, \ldots, \) is a vector of complex-valued coefficients;

- **Karhunen-Loève decomposition**
  \( W \) is a matrix of orthogonal basis and the coefficients \( \gamma \) are independent Gaussian random variables, \( \gamma \sim N(0, \Lambda) \), where and \( \Lambda = \text{diag}(\lambda_1, \ldots, \lambda_n) \).
**Multiresolution representation** (Matsuo et al., 2008) In this work

1. $W$ is a multi-resolution or wavelet basis
2. the coefficients may not be uncorrelated, $\gamma \sim N(0, D)$, where $D$ can be not orthogonal
3. Because of the **localized support** of wavelet basis functions, the expansion results in a small number of coefficients with significant correlations.
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Spatial model

Let $y$ be the field values on a large (regular) 2-D $(N \times N)$-grid (stacked as a vector) with covariance function

$$\Sigma = COV(y).$$

- Eigen decomposition of the covariance

$$\Sigma = WDW^T = WHH^TW^T$$

where $D = H^2$ and $H = (W^{-1}\Sigma W^T)^{1/2}$.

- Representation of the process

$$y = WH\gamma$$

where $\gamma$ is a vector of independent standard normal variables

$W \rightarrow$ need not be orthogonal!

$D \rightarrow$ need not be diagonal!
The matrix $W$ (Kwong and Tang, 1994)
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Observational data

Suppose that the observations $y$ are samples of a centered Gaussian random field and are composed of two components: the observations at irregularly distributed locations, $y_0$, and the missing observations, $y_m$,

$$y = \begin{pmatrix} y_0 \\ y_m \end{pmatrix} \quad (1)$$

The observational model can be written as

$$y_0 = Ky + \varepsilon$$

where

- $y_0$ observations
- $y$ values on a the grid
- $K$ is a incidence matrix
- $\varepsilon \sim MN(0, \sigma^2 I)$
The conditional distribution of $y_m$ given $y_o$ is a multivariate normal with mean

$$
\Sigma_{o,m}(\Sigma_{o,o})^{-1}y_o
$$

and variance

$$
\Sigma_{m,m} - \Sigma_{m,o}(\Sigma_{o,o})^{-1}\Sigma_{o,m}
$$

where

- $\Sigma_{o,m} = W_oHH^T W_m^T$ is the cross-covariance between observed and missing data,
- $\Sigma_{o,o} = W_oHH^T W_o^T + \sigma^2 I$ is covariance of observed data and
- $\Sigma_{m,m} = W_mHH^T W_m^T$ is the covariance of missing data.
- The matrices $W_o$ and $W_m$ are wavelet basis evaluated at the observed and missing data, respectively.

**Problem**

$\Sigma_{o,m}$ and $\Sigma_{o,o}$ very big!!!
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The reduced rank covariance for $\Sigma$ (Nicolis and Nychka, 2012).

- The idea is to estimate $H$ on a small sub-grid $G$ of size $(g \times g)$ starting from a Matérn model and using a MR approach.
- Monte Carlo conditional simulation provide an efficient estimator for the conditional mean and variance.
Conditional simulation

1. Find Kriging prediction on the grid $G$:

   $\hat{y}_g = \Sigma_{o,g}(\Sigma_{o,o})^{-1} y_o,$

   where $\tilde{H}_g = (W_g^{-1} \Sigma_{g,g} W_g^T)^{1/2}$ and $\Sigma_{g,g}$ is stationary covariance model (e.g., Matern).

2. Generate synthetic "data": $y^s$ from $y^s = W\tilde{H}_g a$ with $a \sim N(0, 1)$.

3. Simulated Kriging error:

   $u^* = y_g - y^s_g,$

   where $y_g = W_g\tilde{H}_g a$ and $y^s_g = \Sigma_{o,g}(\Sigma_{o,o})^{-1} y^s_o.$

4. Find conditional field $y_m | y_o$:

   $\hat{y}_u = \hat{y}_g + u^*.$

5. Compute the conditional covariance on $T$ replications, $\Sigma_u = COV(\hat{y}_u)$, using the new $\tilde{H}_g$ in the step 1 of each iteration.
By choosing properly the filter $W$ basis and the levels of resolutions $L$ we obtain the estimation of the conditional mean and variance

$$\Sigma_{o,m}(\Sigma_{o,o})^{-1}y_o \quad (4)$$

and variance

$$\Sigma_{m,m} - \Sigma_{m,o}(\Sigma_{o,o})^{-1}\Sigma_{o,m} \quad (5)$$

where

$$\Sigma_{o,m} = W_o H_g H_g^T W_m, \quad \Sigma_{o,o} = W_o H_g H_g^T W_o + \sigma^2 I \quad \text{and}$$

$$\Sigma_{m,m} = W_m H_g H_g^T W_m$$
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Non-stationary random field simulated on a $40 \times 40$ grid with Matèrn covariance ($\theta = 0.1$ and $\nu = 0.5$)) and 50% of missing values.
Application to AOT (54 × 32)

Reduced Rank Covariance

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REDUCED RANK SPATIAL COVARIANCE
Non-stationary covariance obtained after 5 iterations of MC simulations.
Daily max 8 hour ozone, June 18, 1987

(a)

(b)
Daily NO2 in California
Covariance NO2 in California
Forecasting NO2 in California
Further work

- Find a parametrization for the matrix $H_g$, depending on the parameters of the Matern covariance, $H_g(\nu, \theta)$, and find the maximum likelihood estimates.
- Consider other basis functions (frames, radial basis etc.).
- Include the multiresolution covariances in spatio-temporal models.
- Extension to multivariate case (for example calibration of aerosol data)

