**Exam Cheat Sheet**

**Binomial Coefficient:** For \( r \leq n \),
\[
\binom{n}{m} = \frac{n!}{m!(n-m)!}
\]
represents the number of possible combinations of \( n \) objects taken \( m \) at a time.

**Multinomial Coefficient:** For \( n_1 + n_2 + \cdots + n_r = n \),
\[
\binom{n}{n_1, n_2, ..., n_r} = \frac{n!}{n_1!n_2!\cdots n_r!}
\]
represents the number of possible ways \( n \) objects can be divided into \( r \) groups with \( n_1, n_2, ..., n_r \) objects in each, respectively.

**Axioms for Probability Measures:** The following 3 axioms define a probability measure:

1. For all events \( E \), \( 0 \leq P(E) \leq 1 \).
2. If \( S \) is the sample space, \( P(S) = 1 \).
3. For any sequence of mutually exclusive events \( E_1, E_2, ..., E_n \),
\[
P(\bigcup_{i=1}^{\infty} E_i) = \sum_{i=1}^{\infty} P(E_i).
\]

**Useful Formulas for Probability Measures:**

\[
P(E^c) = 1 - P(E)
\]
\[
P(E) \leq P(F) \text{ if } E \subseteq F
\]
\[
P(E) = P(EF) \text{ if } E \subseteq F
\]
\[
P(E \cup F) = P(E) + P(F) - P(EF)
\]
\[
P(E) = P(EF) + P(EF^c)
\]
\[
P(E \cup F \cup G) = P(E) + P(F) + P(G) - P(EF) - P(EG) - P(FG) + P(EFG)
\]

**Conditional Probability:** The probability of event \( E \), given that we observe \( F \), is:
\[
P(E \mid F) = \frac{P(EF)}{P(F)}
\]
Also, \( P(E \mid F) \) defines a new probability measure with \( F \) being the restricted sample space.

**Independence:** Two items are independent if \( P(EF) = P(E)P(F) \). If \( P(E \mid F) = P(E) \), \( E \) and \( F \) are independent; when \( P(F) \neq 0 \), the two definitions are exactly equivalent.

**Useful Formulas for Contitional Probability:**
\[
P(EF) = P(E \mid F)P(F) = P(F \mid E)P(E)
\]
\[
P(E) = P(E \mid F)P(F) + P(E \mid F^c)P(F^c)
\]
\[
P(E_1E_2\cdots E_n) = P(E_1)P(E_2 \mid E_1)P(E_3 \mid E_2E_1)\cdots P(E_n \mid E_{n-1}E_{n-2}\cdots E_1)
\]
\[
= P(E_n)P(E_{n-1} \mid E_n)P(E_{n-2} \mid E_{n-1}E_n)\cdots P(E_1 \mid E_2E_3\cdots E_n)
\]
\[
P(E \mid F) = \frac{P(F \mid E)P(E)}{P(F)}
\]
**Expectation – Discrete:** Suppose $p(x) = P(X = x)$ is the probability mass function of a R.V. $X$. Then

$$E[X] = \sum_x x p(x)$$

$$E[g(X)] = \sum_x g(x) p(x)$$

$$\text{Var}[X] = E[(X - \mu)^2] = E[X^2] - (E[X])^2,$$

where $\mu = E[X]$

**Discrete Distributions:**

If $X \sim \text{Bernoulli}(x; p)$, where $p$ is the probability of success, then

$$P(X = x) = \begin{cases} p & x = 1 \\ 1 - p & x = 0 \\ 0 & \text{otherwise} \end{cases}, \quad E[X] = p, \quad \text{Var}[X] = p(1 - p)$$

If $X \sim \text{Binomial}(x; n, p)$, where $p$ is the prob. of success and $n$ is the number of tries, then

$$P(X = x) = \begin{cases} \binom{n}{x} p^x (1 - p)^{n-x} & x \in \{0, 1, ..., n\} \\ 0 & \text{otherwise} \end{cases}, \quad E[X] = np, \quad \text{Var}[X] = np(1 - p)$$

If $X \sim \text{Geometric}(x; p)$, then

$$P(X = x) = \begin{cases} (1 - p)^{x-1} p & x \in \{1, 2, \ldots\} \\ 0 & \text{otherwise} \end{cases}, \quad E[X] = \frac{1}{p}, \quad \text{Var}[X] = \frac{1 - p}{p^2}$$

If $X \sim \text{NegativeBinomial}(x; r, p)$, where $p$ is the prob. of success, and $r$ is the number of successes required,

$$P(X = x) = \begin{cases} (r-1)^{-1} p^r (1 - p)^{x-r} & x \in \{r, r+1, \ldots\} \\ 0 & \text{otherwise} \end{cases}, \quad E[X] = \frac{r}{p}$$

If $X \sim \text{Poisson}(x; \lambda)$, where $\lambda$ is the rate parameter, then

$$P(X = x) = \begin{cases} \frac{e^{-\lambda} \lambda^x}{x!} & x \in \{0, 1, 2, \ldots\} \\ 0 & \text{otherwise} \end{cases}, \quad E[X] = \lambda, \quad \text{Var}[X] = \lambda$$

If $X \sim \text{HyperGeometric}(x; n, N, m)$, then $X$ represents the number of red balls in a sample of $n$ balls drawn from an urn with $N$ balls, $m$ of which are red. Then

$$P(X = x) = \begin{cases} \frac{\binom{x}{m} \binom{N-m}{n-x} \binom{N}{n}}{\binom{N}{n}} & x \in \{0, 1, \ldots, n\} \\ 0 & \text{otherwise} \end{cases}, \quad E[X] = \frac{nm}{N}$$
Expectation: Continuous case. Suppose \( f_X(x) \) is the probability density function of \( X \). Then

\[
E[X] = \int_{-\infty}^{\infty} x f_X(x) \, dx
\]

\[
E[g(X)] = \int_{-\infty}^{\infty} g(x) f_X(x) \, dx
\]

\[
\text{Var}[X] = E[(X - \mu)^2] = E[X^2] - (E[X])^2,
\]

where \( \mu = E[X] \)

Continuous distributions:

If \( X \sim \text{Uniform}(x; a, b) \), with \( a < b \), then

\[
f_X(x) = \begin{cases} \frac{1}{b-a} & x \in [a, b] \\ 0 & \text{otherwise} \end{cases}
\]

\[
E[X] = \frac{b+a}{2} \quad \text{Var}[X] = \frac{(b-a)^2}{12}
\]

If \( X \sim \text{Exponential}(x; \lambda) \), with \( \lambda \) being the rate, then

\[
f_X(x) = \begin{cases} \lambda e^{-\lambda x} & x \geq 0 \\ 0 & \text{otherwise} \end{cases}
\]

\[
E[X] = \frac{1}{\lambda} \quad \text{Var}[X] = \frac{1}{\lambda^2}
\]

If \( X \sim \mathcal{N}(x; \mu, \sigma^2) \), where \( \mu \) is the expectation and \( \sigma^2 \) is the variance, then

\[
f_X(x) = \frac{1}{\sqrt{2\pi \sigma}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}
\]

\[
E[X] = \mu \quad \text{Var}[X] = \sigma^2
\]

If \( X \sim \Gamma(x; \alpha, \lambda) \), where \( \alpha \) is the shape parameter and \( \lambda \) is the rate, then

\[
f_X(x) = \begin{cases} \left(\frac{\lambda x}{\Gamma(\alpha)}\right)^{\alpha-1} (\lambda e^{-\lambda x}/\Gamma(\alpha)) & x \geq 0 \\ 0 & x < 0 \end{cases}
\]

\[
E[X] = \frac{\alpha}{\lambda} \quad \text{Var}[X] = \frac{\alpha}{\lambda^2}
\]

If \( X \sim \text{Beta}(x; \alpha, \beta) \), then

\[
f_X(x) = \begin{cases} \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} x^{\alpha-1} (1-x)^{\beta-1} & 0 \leq x \leq 1 \\ 0 & \text{otherwise} \end{cases}
\]

\[
E[X] = \frac{\alpha}{\alpha+\beta}
\]

Standard Normal CDF: If \( Z \sim \mathcal{N}(z; 0, 1) \), then define \( \Phi(z) = P(Z \leq z) \).

Poisson Approximation Theorem: Let \( X \sim \text{Binomial}(x; n, p) \). For \( np \) small relative to \( n \),

\[
P(X = x) \approx P(Y = x), \quad \text{where } Y \sim \text{Poisson}(y; \lambda = np).
\]

DeMoivre-Laplace Approximation Theorem: Let \( X \sim \text{Binomial}(x; n, p) \). For \( np(1-p) \) sufficiently large,

\[
P \left( \frac{X - np}{\sqrt{np(1-p)}} \leq z \right) \approx P(Z \leq z) = \Phi(z)
\]

where \( Z \sim \mathcal{N}(z; 0, 1) \).