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Computing the covariance of two Brownian area integrals

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We compute the expected product of two correlated Brownian area integrals, a problem that arises in the analysis of a popular sorting algorithm. Along the way we find three different formulas for the expectation of the product of the absolute values of two standard normal random variables with correlation θ. These two formulas are found: (a) via conditioning and the non-central chi-square distribution; (b) via Mehler's formula; (c) by representing the correlated normal random variables in terms of independent normal's and integration using polar coordinates.

Key Words and Phrases: bivariate normal distribution, Brownian bridge, correlation, expectation, Mehler's formula, non-central chi-square, product of absolute values.

1 The problem

Suppose that B_j, $j = 1, 2, 3$ are independent Brownian bridge processes on [0, 1]; recall that a Brownian bridge process B is a mean zero Gaussian process on [0, 1] with covariance

$$\mathrm{Cov}(B(s), B(t)) = s \wedge t - st, \quad s, t \in [0, 1].$$

Define two random variables A_1 and A_2 by

$$A_1 = \int_0^1 |B_1(t) - B_2(t)|\,\mathrm{d}t, \quad A_2 = \int_0^1 |B_1(t) - B_3(t)|\,\mathrm{d}t.$$

The following question arose in the course of trying to analyze the behavior of a certain sorting algorithm; see SMYTHE and WELLNER (1999):

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Problem. What are the numbers $E(A_1A_2)$, $\mathrm{Cov}(A_1, A_2)$, and $\mathrm{Correl}(A_1, A_2)$?

PROPOSITION 1. $E(A_1A_2) \approx .205145\ldots$; $\mathrm{Cov}(A_1, A_2) \approx .008796$; and $\mathrm{Correl}(A_1, A_2) \approx .2378$.

PROOF, PART 1. Note that by Fubini's theorem we have

$$\begin{aligned} E(A_1A_2) &= \int_0^1\int_0^1 E\{|B_1(s) - B_2(s)||B_1(t) - B_3(t)|\}\,\mathrm{d}s\,\mathrm{d}t \\ &= \int_0^1\int_0^1 E\{|Z_1(s)||Z_2(t)|\}\,\mathrm{d}s\,\mathrm{d}t \end{aligned} \tag{1}$$

where $Z_1(s) \equiv B_1(s) - B_2(s), Z_2(t) \equiv B_1(t) - B_3(t)$ have a bivariate normal distribution with zero means, variances $2s(1-s)$ and $2t(1-t)$ respectively, and correlation

$$\mathrm{Corr}(Z_1(s), Z_2(t)) = \left\{ \begin{array}{ll} (1/2)\sqrt{s/t}\sqrt{(1-t)/(1-s)} & \text{if } s \leqslant t \\ (1/2)\sqrt{t/s}\sqrt{(1-s)/(1-t)} & \text{if } t \leqslant s \end{array} \right\} \equiv \theta(s, t). \tag{2}$$

Thus, by standardizing $Z_1(s)$ and $Z_2(t)$, the right side of (1) is equal to

$$\int_0^1\int_0^1 \sqrt{2s(1-s)}\sqrt{2t(1-t)}E\{|\tilde{Z}_1(s)|\ |\tilde{Z}_2(t)|\}\,\mathrm{d}s\,\mathrm{d}t \tag{3}$$

where $\tilde{Z}_1(s)$ and $\tilde{Z}_2(t)$ have a joint normal distribution with 0 means, variances 1, and correlation given by (2). Thus we need to compute $E_\theta(|X|\ |Y|)$ as a function of θ where (X, Y) have a bivariate normal distribution with means 0, variances 1, and correlation θ.

2 Formulas for $E_\theta(|X|\ |Y|)$

Here we prove the following proposition giving two different infinite series representations of the function $E_\theta(|X|\ |Y|)$.

PROPOSITION 2. *If* (X, Y) *have a bivariate normal distribution with means* 0, *variances* 1, *and correlation* θ, *then*

$$E_\theta(|X|\ |Y|) = \frac{2}{\sqrt{\pi}}(1-\theta^2)^{3/2}\sum_{k=0}^{\infty}\theta^{2k}\frac{\Gamma(k+1)}{\Gamma(k+1/2)} \tag{4}$$

$$= \frac{1}{2\pi}\left\{4 + \sum_{k=1}^{\infty}\theta^{2k}(2k)!\left[\sum_{j=0}^{k}(-1)^j\frac{2^{k-j+1}}{j!2^j}\frac{(k-j)!}{(2k-2j)!}\right]^2\right\} \tag{5}$$

$$= \frac{2}{\pi}\{\theta\arcsin(\theta) + \sqrt{1-\theta^2}\}. \tag{6}$$

PROOF. We will show that (4), (5), and (6) are all valid formulas representing the expectation $E_\theta(|X||Y|)$ as a function of θ. The first series formula is obtained by conditioning and using the series representation for the non-central chi-square distribution, while the second series formula is obtained by using Mehler's formula for the Radon–Nikodym derivative of the bivariate normal distribution with correlation θ with respect to the normal distribution with correlation 0 (independence). The third expression is obtained by representing (X, Y) in terms of independent $N(0, 1)$ random variables, and integrating via use of polar coordinates.

To prove (4), first write $E(|X||Y|) = E(|X|E(|Y|\big|X))$ where the conditional distribution of Y given X is $N(\theta X, 1-\theta^2)$. Thus the distribution of $Y/\sqrt{1-\theta^2}$ conditionally on X is $N(\theta X/\sqrt{1-\theta^2}, 1)$. Hence it follows (see e.g. GRAYBILL 1976, 125–126) that

$$\frac{|Y|}{\sqrt{1-\theta^2}} = \sqrt{\frac{Y^2}{1-\theta^2}} \sim \sqrt{\chi_1^2\left(\frac{\theta^2 X^2}{1-\theta^2}\right)}$$

where $Z \sim \chi_1^2(\delta)$ has density

$$f_Z(z) = \sum_{k=0}^{\infty} p_k(\delta/2) g(z; (2k+1)/2, 1/2);$$

here $p_k(\delta/2)$ is the Poisson $(\delta/2)$ probability given by

$$p_k(\delta/2) = \mathrm{e}^{-\delta/2}\frac{(\delta/2)^k}{k!},$$

and $g_k(z; (2k+1)/2, 1/2)$ is the Gamma density with parameters $(2k+1)/2$ and $1/2$ given by

$$g_k(z; (2k+1)/2, 1/2) = \frac{1}{2}\frac{(z/2)^{\frac{(2k+1)}{2}-1}}{\Gamma(k+1/2)}\exp(-z/2).$$

Thus

$$\begin{aligned} EZ^{1/2} &= \sum_{k=0}^{\infty} p_k(\delta/2)\int_0^{\infty} z^{1/2}\frac{1}{2}\frac{(z/2)^{\frac{(2k+1)}{2}-1}}{\Gamma(k+1/2)}\exp(-z/2)\,\mathrm{d}z \\ &= \sqrt{2}\sum_{k=0}^{\infty} p_k(\delta/2)\frac{\Gamma(k+1)}{\Gamma(k+1/2)}. \end{aligned}$$

Hence we find that

$$E_\theta(|X||Y|) = \sqrt{1-\theta^2}E\left\{|X|E\left[\frac{|Y|}{\sqrt{1-\theta^2}}|X\right]\right\}$$

$$= \sqrt{2(1-\theta^2)} E\left\{ |X| \sum_{k=0}^{\infty} \exp\left(-\frac{\theta^2 X^2}{2(1-\theta^2)} \right) \frac{(\theta^2 X^2/2(1-\theta^2))^k}{k!} \frac{\Gamma(k+1)}{\Gamma(k+1/2)} \right\}$$

$$= \sqrt{2(1-\theta^2)} \sum_{k=0}^{\infty} \frac{\Gamma(k+1)}{k!\Gamma(k+1/2)} E\left\{ |X| \exp\left(-\frac{\theta^2 X^2}{2(1-\theta^2)} \right) \left(\frac{(\theta^2 X^2)}{2(1-\theta^2)} \right)^k \right\}$$

$$= \frac{2}{\sqrt{\pi}} (1-\theta^2)^{3/2} \sum_{k=0}^{\infty} \theta^{2k} \frac{\Gamma(k+1)}{\Gamma(k+1/2)},$$

after computing the expectation under the sum and finding that it equals

$$\sqrt{2/\pi}\theta^{2k}(1-\theta^2)\Gamma(k+1).$$

Thus (4) holds. (We call (4) *Perlman's formula* since this method was suggested to us by Michael Perlman).

Now for (5). Mehler's formula (see e.g. MEHLER, 1866, BUJA, 1990, and KLAASSEN and WELLNER, 1997, 74) says that the bivariate normal density ϕ_θ of (X, Y) can be written as

$$\phi_\theta(x, y) = \left\{ 1 + \sum_{k=1}^{\infty} \theta^k H_k^*(x) H_k^*(y) \right\} \phi(x)\phi(y);$$

here $\phi(x) = (2\pi)^{-1/2} \exp(-x^2/2)$ is the standard normal density, and $\{H_k^*\}$ are the Hermite polynomials normalized to be orthonormal with respect to ϕ. Since $E|X| = E|Y| = \sqrt{2/\pi}$, by interchanging the sum and the integration we find that

$$E_\theta(|X||Y|) = \frac{2}{\pi} + \sum_{k=1}^{\infty} \theta^k b_k^2$$

where

$$b_k \equiv \int |x| H_k^*(x)\phi(x)\,\mathrm{d}x = \frac{1}{\sqrt{2\pi k!}} \int |x| H_k(x) \exp(-x^2/2)\,\mathrm{d}x \equiv \frac{1}{2\pi k!} c_k.$$

Now

$$H_k(x) = k! \sum_{j=0}^{[k/2]} (-1)^j \frac{1}{j! 2^j (k-2j)!} x^{k-2j};$$

see e.g. ABRAMOWITZ and STEGUN (1972), 774, where their He_n is our H_n. Hence we compute $c_k = 0$ for k odd, and for k even

$$c_k = k! \sum_{j=0}^{[k/2]} (-1)^j \frac{1}{j! 2^j (k-2j)!} \int_{-\infty}^{\infty} |x| x^{k-2j} \mathrm{e}^{-x^2/2}\,\mathrm{d}x$$

$$= k! \sum_{j=0}^{[k/2]} (-1)^j \frac{2^{(k/2)-2j+1}}{j!} \frac{(k/2-j)!}{(k-2j)!}.$$

For example, Mathematica yields $c_2 = 2,\ c_4 = -2,\ c_6 = 6,\ c_8 = -30,\ c_{10} = 210, \ldots$ Combining the above yields

$$E_\theta(|X||Y|) = \frac{1}{2\pi}\left\{4 + \sum_{k=1}^{\infty} \theta^{2k}(2k)!\left\{\sum_{j=0}^{k}(-1)^j \frac{2^{k-j+1}}{j!2^j}\frac{(k-j)!}{(2k-2j)!}\right\}^2\right\}$$
$$= \frac{1}{2\pi}\left\{4 + \sum_{k=1}^{\infty} \theta^{2k}\frac{c_{2k}^2}{(2k)!}\right\}.$$

Thus (5) holds. (We call (5) *Wellner's formula* because the first author obtained it via Mehler's formula in the course of this study.)

To prove (6), note that if X and Z are independent $N(0, 1)$, then defining

$$Y \equiv \theta X + \sqrt{1-\theta^2}Z,$$

it follows that (X, Y) has a bivariate normal distribution with zero means, unit variances, and correlation θ. Thus it follows by using polar coordinates for the joint distribution of (X, Z), that

$$E|XY| = \frac{1}{2\pi}\int_0^\infty \int_0^{2\pi} |\theta r^2 \cos^2\phi + \sqrt{1-\theta^2}r^2 \cos\phi \sin\phi| r \mathrm{e}^{-r^2/2}\,\mathrm{d}r\,\mathrm{d}\phi$$
$$= \frac{1}{2\pi}\int_0^\infty r^3 \mathrm{e}^{-r^2/2}\,\mathrm{d}r \int_0^{2\pi} \cos^2\phi |\theta + \sqrt{1-\theta^2}\tan\phi|\,\mathrm{d}\phi$$
$$= \frac{2}{\pi}\{\theta \arcsin(\theta) + \sqrt{1-\theta^2}\}$$

where the last equality follows by noting that the function $\phi \mapsto \theta + \sqrt{1-\theta^2}\tan\phi$ is positive for $\phi \in (0, \pi/2) \cup (\phi_1(\theta), 3\pi/2) \cup (\phi_2(\theta), 2\pi)$, negative for $\phi \in (\pi/2, \phi_1(\theta)) \cup (3\pi/2, \phi_2(\theta))$, where $\phi_1(\theta) \equiv \arctan(-\theta/\sqrt{1-\theta^2}) \in (\pi/2, \pi)$, $\phi_2(\theta) \equiv \phi_1(\theta) + \pi$, and then doing a careful integration for the separate pieces. (We call (6) the *Janson–Smythe formula* because it was derived by the second author after a suggestion by S. Janson.) □

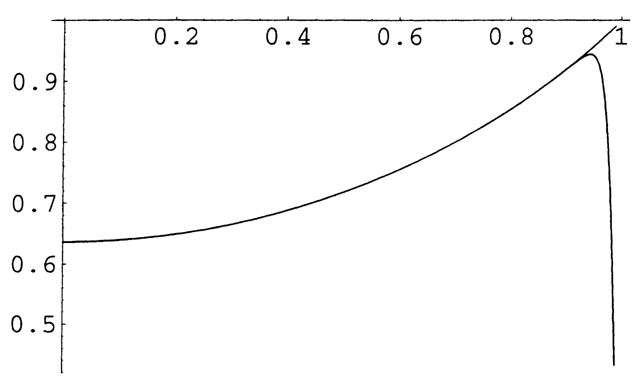

Fig. 1. Plot of $E_\theta|X|\,|Y|$.

In the above plot, the graph descending rapidly near 1 is the plot via Perlman's formula; the graph which increases monotonically to 1 is from Wellner's formula and the Janson–Smythe formula. The divergence near 1 is due to truncation of the series for g at 50 terms. The plot was produced by Mathematica code available from the first author.

Added in proof: Steve Stigler has pointed out that (6) (and other absolute moments of bivariate and trivariate normal distributions) were derived by Nabeya (1951) and Kamat (1953); also see Johnson and Kotz (2000), page 91 and (13) on page 92. In particular, our (6) is given by Kamat (1953), page 26, in his formula for (1,1,0).

3 Completion; combining the parts

Proof of Proposition 1, Part 2. Now we can use the series (4) and (5) of Proposition 2 to compute the expectation inside the integral in (3). By the series in (4)

$$E\{|\tilde{Z}_1(s)||\tilde{Z}_2(t)|\} = \frac{2}{\sqrt{\pi}}(1-\theta^2(s,t))^{3/2}\sum_{k=0}^{\infty}\theta^{2k}(s,t)\frac{\Gamma(k+1)}{\Gamma(k+1/2)}$$

where $\theta(s,t)$ is given by (2). Plugging this in and changing the order of summation and integration we find that $E(A_1A_2)$ is equal to

$$\int_0^1\int_0^1 \sqrt{2s(1-s)}\sqrt{2t(1-t)}\frac{2}{\sqrt{\pi}}(1-\theta^2(s,t))^{3/2}\sum_{k=0}^{\infty}\theta^{2k}(s,t)\frac{\Gamma(k+1)}{\Gamma(k+1/2)}\,\mathrm{d}s\,\mathrm{d}t$$
$$=\frac{4}{\sqrt{\pi}}\sum_{k=0}^{\infty}\frac{\Gamma(k+1)}{\Gamma(k+1/2)}\int_0^1\int_0^1 \sqrt{s(1-s)}\sqrt{t(1-t)}(1-\theta^2(s,t))^{3/2}\theta^{2k}(s,t)\,\mathrm{d}s\,\mathrm{d}t$$
$$\equiv\frac{4}{\sqrt{\pi}}\sum_{k=0}^{\infty}\frac{\Gamma(k+1)}{\Gamma(k+1/2)}b_k$$

Computing the integrals numerically in Mathematica yields Table 1.

Table 1. Values of the integrals b_k

k	b_k
0	0.134498
1	0.0108539
2	0.00146832
3	0.000245253
4	4.56330×10^{-5}
5	9.05168×10^{-6}
6	1.87238×10^{-6}
7	3.98809×10^{-7}
8	8.68010×10^{-8}
9	1.92028×10^{-8}
10	4.30522×10^{-9}

Using the numerical values of b_k to compute the series, we find that $E(A_1A_2) \approx .205145$.

Alternatively, by the series in (5),

$$E\{|\tilde{Z}_1(s)||\tilde{Z}_2(t)|\} = \frac{1}{2\pi}\left\{4 + \sum_{k=1}^{\infty} \theta^{2k}(s,t)\frac{c_{2k}^2}{(2k)!}\right\},$$

and plugging this in and changing the order of summation and integration, we find that $E(A_1A_2)$ is equal to

$$\begin{aligned}
&\int_0^1\int_0^1 \sqrt{2s(1-s)}\sqrt{2t(1-t)}\frac{1}{2\pi}\left\{4 + \sum_{k=1}^{\infty}\theta^{2k}(s,t)\frac{c_{2k}^2}{(2k)!}\right\}\mathrm{d}s\,\mathrm{d}t \\
&= \frac{1}{2\pi}\left\{8\int_0^1\int_0^1 \sqrt{s(1-s)}\sqrt{t(1-t)}\,\mathrm{d}s\,\mathrm{d}t\right. \\
&\qquad\left. + 2\sum_{k=1}^{\infty}\frac{c_{2k}^2}{(2k)!}\int_0^1\int_0^1 \sqrt{s(1-s)}\sqrt{t(1-t)}\theta^{2k}(s,t)\,\mathrm{d}s\,\mathrm{d}t\right\} \\
&= \frac{1}{2\pi}\left\{8\left(\frac{\pi}{8}\right)^2 + 2\sum_{k=1}^{\infty}\frac{c_{2k}^2}{(2k)!}d_{2k}\right\} \\
&= \frac{\pi}{16} + \frac{1}{\pi}\sum_{k=1}^{\infty}\frac{c_{2k}^2}{(2k)!}d_{2k}.
\end{aligned}$$

Evaluating the series numerically we find again (with only four terms) that $E(A_1A_2) \approx .205145\ldots$.

Using the third formula (6), we find that

$$\begin{aligned}
E(A_1A_2) &= \int_0^1\int_0^1 \sqrt{2s(1-s)}\sqrt{2t(1-t)}\frac{2}{\pi} \\
&\qquad \times \{\theta(s,t)\arcsin(\theta(s,t)) + \sqrt{1-\theta^2(s,t)}\}\,\mathrm{d}s\,\mathrm{d}t \\
&= \frac{8}{\pi}\int_0^1\int_0^t \sqrt{s(1-s)t(1-t)} \\
&\qquad \times \left\{\frac{1}{2}\sqrt{\frac{s}{t}\frac{1-t}{1-s}}\arcsin\left(\frac{1}{2}\sqrt{\frac{s}{t}\frac{1-t}{1-s}}\right) + \sqrt{1-\frac{1}{4}\frac{s}{t}\frac{1-t}{1-s}}\right\}\mathrm{d}s\,\mathrm{d}t \\
&= \frac{4}{\pi}\int_0^1\int_0^t s(1-t)\arcsin\left(\frac{1}{2}\sqrt{\frac{s}{t}\frac{1-t}{1-s}}\right)\mathrm{d}s\,\mathrm{d}t \\
&\qquad + \frac{8}{\pi}\int_0^1\int_0^t \sqrt{s(1-s)t(1-t)}\sqrt{1-\frac{1}{4}\frac{s}{1-s}\frac{1-t}{t}}\,\mathrm{d}s\,\mathrm{d}t \\
&\equiv \frac{8}{\pi}\left\{\frac{1}{2}I + II\right\}.
\end{aligned}$$

Now by letting $x \equiv s/(1-s)$ and $y \equiv t/(1-t)$,

Table 2. Values of the integrals c_{2k} and d_{2k}

k	c_{2k}	d_{2k}
1	2	0.0136449
2	−2	0.00194758
3	6	0.00033606
4	−30	6.38754×10^{-5}
5	210	1.28602×10^{-5}
6	−1890	2.68914×10^{-6}
7	270270	5.77651×10^{-7}
8	−270270	1.26556×10^{-7}

$$
\begin{aligned}
II &= \int_0^1\int_0^1 \sqrt{s(1-s)t(1-t)}\sqrt{1-\frac{1}{4}\frac{s}{1-s}\frac{1-t}{t}}\,\mathrm{d}s\,\mathrm{d}t \\
&= \int_0^\infty\int_0^y \frac{\sqrt{xy}}{(1+x)^3(1+y)^3}\sqrt{1-\frac{1}{4}\frac{x}{y}}\,\mathrm{d}x\,\mathrm{d}y \\
&= \int_0^\infty \frac{y^2}{(1+y)^3}\int_0^1 \frac{u^{1/2}}{(1+yu)^3}\sqrt{1-u/4}\,\mathrm{d}u\,\mathrm{d}y \\
&= \int_0^1 u^{1/2}\sqrt{1-u/4}\int_0^\infty \frac{1}{(1+uy)^3}\frac{y^2}{(1+y)^3}\,\mathrm{d}y\,\mathrm{d}u
\end{aligned}
$$

On the other hand,

$$
\begin{aligned}
I &= \int_0^1\int_0^t s(1-t)\arcsin\left(\frac{1}{2}\sqrt{\frac{s}{t}\frac{1-t}{1-s}}\right)\mathrm{d}s\,\mathrm{d}t \\
&= \int_0^1 t(1-t)\int_0^t \frac{s}{t}\arcsin\left(\frac{1}{2}\sqrt{\frac{s}{t}\frac{1-t}{1-s}}\right)\mathrm{d}s\,\mathrm{d}t \\
&= \int_0^\infty \frac{1}{(1+y)^3}\int_0^y \frac{x}{(1+x)^3}\arcsin\left(\frac{1}{2}\sqrt{x/y}\right)\mathrm{d}x\,\mathrm{d}y \\
&= \int_0^\infty \frac{y^2}{(1+y)^3}\int_0^1 \frac{u}{(1+yu)^3}\arcsin\left(\frac{1}{2}\sqrt{u}\right)\mathrm{d}u\,\mathrm{d}y \\
&= \int_0^1 u\arcsin\left(\frac{1}{2}\sqrt{u}\right)\int_0^\infty \frac{y^2}{(1+y)^3(1+yu)^3}\,\mathrm{d}y\,\mathrm{d}u.
\end{aligned}
$$

Combining these pieces yields

$$
E(A_1A_2) = \frac{8}{\pi}\int_0^1 \left\{\frac{1}{2}u\arcsin(\sqrt{u}/2) + \sqrt{u(1-u/4)}\right\}k(u)\,\mathrm{d}u
$$

where

$$
k(u) \equiv \int_0^\infty \frac{y^2}{(1+y)^3(1+yu)^3}\,\mathrm{d}y = \frac{\log(1/u)\{u^2+4u+1\}+3u^2-3}{(1-u)^5}
$$

satisfies $\lim_{u \to 1} k(u) = 1/30$. Evaluating these integrals numerically (by taking $k(u) = 1/30$ for $.995 \leqslant u \leqslant 1$), yields $E(A_1 A_2) = 0.205143$.

It follows from results of SHEPP (1992) (see also PERMAN and WELLNER, 1996, 1107) that $E(A_1) = E(A_2) = \sqrt{\pi}/4 \approx .4431\ldots$ and $\text{Var}(A_1) = \text{Var}(A_2) = 7/30 - \pi/16 \approx .03698\ldots$. This results in $\text{Cov}(A_1, A_2) \approx .008795$, and $\text{Correl}(A_1, A_2) \approx .2378$. □

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