Last time we arrived at the Central Limit Theorem (CLT):

**Strong Version:** If $X \sim$ any dist. with mean $= \mu_x$, var. $= \sigma_x^2$

Then $\bar{X} \sim N(\mu_x, \frac{\sigma_x}{\sqrt{n}})$ for large $n$.

which allows us to compute probs like

$$\Pr(a < \bar{X} < b) = \Pr\left( \frac{a - \mu_x}{\sigma_x/\sqrt{n}} < \frac{\bar{X} - \mu_x}{\sigma_x/\sqrt{n}} < \frac{b - \mu_x}{\sigma_x/\sqrt{n}} \right) = \Pr(a < Z < b) = \text{Table I}$$

Compare with Ch. 1 probs, which were like

$$\Pr(a < X < b) = \Pr\left( \frac{a - \mu_x}{\sigma_x} < \frac{X - \mu_x}{\sigma_x} < \frac{b - \mu_x}{\sigma_x} \right) = \Pr(a < Z < b) = \text{Table I}$$

**Specific types of probs. of common interest in statistics are**

$$\Pr(\bar{X} > \bar{x}_{0.05}) \text{ or } \Pr(\bar{X} < \bar{x}_{0.05}) \text{ Recall } \Pr(\bar{X} = \bar{x}_{0.05}) = 0$$

But the above procedure for computing these probs requires knowledge of the pop. mean and std. dev., $\mu_x, \sigma_x$ (look above!)

There are 2 ways of turning the procedure around so that it actually says something about $\mu_x, \sigma_x$.

1) **Confidence Intervals** Ch. 7
2) **Hypothesis Testing (p-values)** Ch. 8

Recall our notation:

<table>
<thead>
<tr>
<th><strong>Statistics</strong></th>
<th><strong>Pop. Parameters</strong></th>
</tr>
</thead>
<tbody>
<tr>
<td>$\bar{X}$ (sample mean) is a point estimate of $\mu_x$ (pop. mean)</td>
<td></td>
</tr>
<tr>
<td>$S$ (&quot;std. dev.)</td>
<td>$\sigma_x$ (&quot;std. dev.)</td>
</tr>
<tr>
<td>$P$ (&quot;prop.)</td>
<td>$\pi_x$ (&quot;prop.)</td>
</tr>
</tbody>
</table>
| $n$ ("size") is not related to pop. size. For $n = \infty$, for us
The 1st way is to build a confidence interval (CI) for $\mu_x$:

The procedure is to start with $P(-1.96 < z < 1.96) = 0.95$ with specific values of $a, b$, and $\text{blah}$. E.g.

$$P \left( -1.96 < z < 1.96 \right) = 0.95$$

$$P \left( -1.96 < \frac{x - \mu}{\sigma} < 1.96 \right) = 0.95$$

$$P \left( -1.96 \frac{s}{\sqrt{n}} < x - \mu < 1.96 \frac{s}{\sqrt{n}} \right) = 0.95$$

Confidence level:

- $x - 1.96 \frac{s}{\sqrt{n}} < \mu_x < x + 1.96 \frac{s}{\sqrt{n}}$
- Not fixed
- Random

\[ 0.95\% \text{ C.I. for } \mu_x: \quad \bar{x} \pm 1.96 \frac{s}{\sqrt{n}} \]

This is a random CI, because $\bar{x}$ is random (how else would it have a sampling dist?!!)

The (observed) 95% C.I. for $\mu_x$ is

\[ \bar{x}_{\text{obs}} \pm 1.96 \frac{s}{\sqrt{n}} \]

1st Interpretation: We are 95% confident that $\mu_x$ is in here.

2nd " : Below.

Often we forget saying "observed."

It's up to you to find out if we're talking about a random CI or the observed CI.
Suppose a sample of size 25 yields $\bar{x}_{obs} = 3$, $s_{obs} = 1$. What can we say about the population mean?

Suppose the population is normal ($\mu_x = 2$, $\sigma_x = 1$). What's the probability of getting an even larger sample mean?

$$\text{prob}(\bar{x} > \bar{x}_{obs}) = \text{prob}(z > \frac{3-2}{\frac{1}{\sqrt{25}}}) = \text{prob}(z > 5) \approx 0.0$$

$\bar{x} > 3$ is unlikely, if $\mu_x = 2$.

Estimate with $s_x$

(observed) 95% CI for $\mu_x$:

$$\bar{x}_{obs} \pm 1.96 \cdot \frac{s_{obs}}{\sqrt{n}}$$

$$3 \pm 1.96 \cdot \frac{1}{\sqrt{25}} = 3 \pm 0.392 = (2.6, 3.4)$$

Interpret: We can be 95% confident that the true mean is in this interval.

Note: That in spite of all the probs, this interp. does not have prob. CIs have (at least) 2 interpretations.

The 2nd one (below) involves the word probability.
Note that in the last step of the derivation of the C.I. for $\mu_x$, I dropped the pr. That is because $pr(\cdots \Rightarrow \mu_x \Rightarrow \cdots)$ does not exist, because $\mu_x$ is fixed, not random. There is a way of squeezing "probability" into the conclusions, but it has to pertain to the random C.I.

We are 95% confident that the pop. mean
is in the interval $\left( \bar{x} \pm 1.96 \frac{s_x}{\sqrt{n}} \right)$.

Equivalent interpretations of C.I.

There is a 95% prob that a random sample will yield a C.I. $\left( \bar{x} \pm 1.96 \frac{s_x}{\sqrt{n}} \right)$ that covers $\mu_x$.

Look at the derivation of C.I.; This is obvious.

$\{ \begin{align*}
\text{Sample 1} & : \quad \bar{x} \\
\text{Sample 2} & : \quad \bar{x} \\
\end{align*} \}$

$\Rightarrow 95\%$ of these intervals cover $\mu_x$.

$\Rightarrow$ i.e. The prob. that a random C.I. $\left( \bar{x} \pm 1.96 \frac{s_x}{\sqrt{n}} \right)$ will include $\mu_x$ is 0.95.

If you want to say something directly about $\mu_x$, use "confidence" not prob.

C.I.'s are all about coverage;
a 95% C.I. for $\mu_x$ is designed to cover $\mu_x$ in 95% of samples.

For the above example: (Observed) 95% CI (2.6, 3.4)
2nd interp.: There is 95% prob. that a random CI will cover $\mu_x$. 
[Q1] For the above e.g., which of the following is correct.

A) The prob that $2.6 < \mu_x < 3.4$ is 95% $\mu_x = \text{fixed}$
B) $2.6 < x_{100} < 3.4$ $\bar{x}_{100} = \text{fixed}$
C) $\mu_x - 1.96 \sigma_{\bar{x}} / \sqrt{n} < \bar{x} < \mu_x + 1.96 \sigma_{\bar{x}} / \sqrt{n}$ Defn. of CI.
D) none of the above.

Note $\Pr (2.6 < \bar{x} < 3.4) \neq 0.95$.

What about other confidence levels ($\neq 0.95$)?

E.g. 99% conf. level: "self-evident fact."

$\Pr(-2.575 < z < 2.575) = 0.99$  \[\text{Table I}\]

\[
\frac{\bar{x} - \mu_x}{\sigma_{\bar{x}} / \sqrt{n}} \Rightarrow \text{C.I. for } \mu_x : \bar{x} \pm 2.575 \frac{\sigma_{\bar{x}}}{\sqrt{n}}
\]

In general: \[\text{C.I. for } \mu_x : \bar{x} \pm z^* \frac{\sigma_{\bar{x}}}{\sqrt{n}} \] "multiplier"

where $z^* = 1.645, 1.96, 2.575, \ldots$

for conf. level = 90%, 95%, 99%, $\ldots = 1 - \alpha$

or $\alpha$-level = 0.1, 0.05, 0.01, $\ldots$

You can either "derive" these $z^*$ values from Table I (just like we did for the above examples), or look them up on the last line of Table IV.
WARNING:

The math is Trivial!

It's the interpretations of CI that are really difficult.

For the previous example: (Observed) 95% CI: (2.6, 3.4)

1) We can be 95% confident that the true mean is in here.

2) There is a 95% probability that a random 95% CI will cover μ.

Note that the 2nd interpretation makes no reference to (2.6, 3.4)!

Relationship between probability and confidence:

→ **Probability** acts on **random** things, like sample means.

  e.g. \( \text{prob}(\bar{x} > 3) \) is perfectly meaningful.

  \( \text{prob}(\mu > 3) \) makes no sense! \{ important \}

→ **Confidence** acts on **fixed** things, like population means.

  e.g. C.I. for \( \mu \) is perfectly meaningful.

  C.I. for \( \bar{x} \) makes no sense!
Suppose you have computed a 95% C.I. for \( \mu_x \) based on a sample of size \( n \). Your friend, however, wants to compute a 99% C.I. for \( \mu_x \). How big should his sample size (\( m \)) be in order for the two C.I.s to have the same width?

(a) It turns out that the sample std. dev., \( s \), has a normal distr. with parameters \( \sigma_x \) and \( \sigma_x/\sqrt{n} \), where \( \sigma_x \) is the pop. std. dev. Now, follow the procedure we have developed, starting from a “self-evident fact” to develop a C.I. formula for \( \sigma_x \).

(b) Suppose for a specific data set based on a sample of size 169, we have found the sample std. dev. of 3.73. Compute the 95% CI for the pop. std. dev.

(c) Provide 2 interpretations.