2.1 Symmetrization

For any function class $F$, we define the empirical Rademacher complexity of $F$ to be:

$$\tilde{R}_n(G) := \mathbb{E}_\epsilon \left( \sup_{f \in F} \left| \frac{1}{n} \sum_{i=1}^{n} \epsilon_i f(x_i) \right| \right)$$

where the expectation is on the Rademacher sequence $\{\epsilon_i, i \in [n]\}$ conditionally on the data $\{x_1, \ldots, x_n\}$.

**Problem 1**

Consider the functional class $F = \{x \rightarrow \text{sign}(\langle \theta, x \rangle) \mid \theta \in \mathbb{R}^d, \|\theta\|_2 = 1\}$, corresponding to the $\{-1, +1\}$-valued classification rules defined by linear functions in $\mathbb{R}^d$. Suppose $d \geq n$ so that it is possible to have $x_n^* = \{x_1, \ldots, x_n\}$ that is a collection of vectors in $\mathbb{R}^d$ that are linearly independent. For such an $x_n^*$, show that

$$\mathbb{E}_\epsilon \left( \sup_{f \in F} \left| \frac{1}{n} \sum_{i=1}^{n} \epsilon_i f(x_i) \right| \right) = 1.$$

2.2 VC dimension

**Problem 2**

Let $A$ be a finite class of sets (i.e., $|A| < \infty$). Determine upper bounds of $\Pi_A(n)$ and $\nu(A)$. Provide an example for which your upper bounds are tight.

**Problem 3**

Determine the VC dimensions of the following classes of sets:

(a) The class of sets in $\mathbb{R}^d$:

$$A := \{(-\infty, a_1] \times (-\infty, a_2] \times \cdots \times (-\infty, a_d] \mid (a_1, \ldots, a_d) \in \mathbb{R}^d\}.$$
(b) The class of sets in $\mathbb{R}^d$:
\[ A := \{(b_1, a_1) \times (b_2, a_2) \times \cdots \times (b_d, a_d) \mid (a_1, \ldots, a_d), (b_1, \ldots, b_d) \in \mathbb{R}^d\}. \]

**Problem 4**

Determine the VC dimensions of the following classes of sets:

(a) A half-space is defined to be a set of the form \( \{x \in \mathbb{R}^d : \langle x, u \rangle \leq c \} \) for some fixed \( u \in \mathbb{R}^d \) and \( c \in \mathbb{R} \). Show that the collection of all half-spaces in \( \mathbb{R}^d \) is a VC-class of index \( d + 1 \).

(b) The class of all closed balls in \( \mathbb{R}^2 \), that is, \( A \) is the class of all subsets of the form
\[ \left\{ x \in \mathbb{R}^2 \mid \sum_{i=1}^{2} (x_i - a_i)^2 \leq R, \text{ for some } (a_1, a_2) \in \mathbb{R}^2 \text{ and } R > 0 \right\}. \]


**2.3 VC-subgraph**

**Problem 5**

Prove the following example in Lecture note 2: Suppose \( C \) is a VC class of index \( \nu(C) \), then by definition the class of functions \( F := \{1_C : C \in C\} \) is VC subgraph of index \( \nu(C) \).

**Problem 6 (Problem 10 on Page 152 in VW1996)**

For a set \( F \) of measurable functions, define “closed” and “open” subgraphs by \( \{(x, t) : t \leq f(x)\} \) and \( \{(x, t) : t < f(x)\} \) respectively. Show that the collections of “closed” and “open” subgraphs have the same VC-index.

**Problem 7 (Problem 20 on Page 153 in VW1996)**

The class of functions of the form \( x \rightarrow c1_{(a,b]}(x) \) with \( a, b, c > 0 \) ranging over \( \mathbb{R} \) is VC-subgraph. Determine the index.

**Problem 8 (Problem 21 on Page 153 in VW1996)**

The “Box-Cox family of transformations” \( F := \{f_\lambda : (0, \infty) \to \mathbb{R} : \lambda \in \mathbb{R} - \{0\}\} \), with \( f_\lambda(x) := (x^\lambda - 1)/\lambda \) is a VC-subgraph class.
Problem 9

Prove Lemma 23 in the lecture note 2.