Believe it or not?

“In poker, don’t forget, if you’re dealt a pair, that automatically increases the odds that your opponent has been dealt a pair, too.” Mike Mailway, Post Intelligencer, 9/26/1992.

Let us agree that “dealing a pair” means that the hand of 5 cards contains exactly one pair and the rest does not add value to the hand, i.e., the other three cards have a face value distinct from the pair face value and their face values are distinct from each other. Let us also assume, that only two hands are dealt, although that does not really matter. We are not stating anything about the odds that at least one of your opponents has a pair. We are talking about a specific opponent and once the first two hands have been dealt it does not matter how the rest of the cards fall.

First we recall how to compute the chance of the event $A$ of getting only a pair in the first hand. There are $\binom{52}{5} = 2,598,960$ ways to get any hand. Now let us count how many hands we can make with exactly one pair. There are 13 different face values that could be the face value of the pair. Once a face value is chosen there are $\binom{4}{2} = 6$ ways to pick the two for a pair from that designated face value. Now we have to choose three cards (to fill the hand) that are of distinct face values. We have to designate 3 face values from the remaining 12, i.e., there are $\binom{12}{3} = 220$ ways of doing that. From the 3 face values we have to pick a card each in $4^3 = 64$ ways. Thus there are $13 \cdot 6 \cdot 220 \cdot 64 = 1,098,240$ of getting such a hand. Thus the probability of this event $A$ is

$$P(A) = \frac{1,098,240}{2,598,960} = .422569.$$  

Let $B$ denote the event that the second hand has exactly a pair. Of course, $P(A) = P(B)$, because any hand that can be dealt as a first hand can also be dealt as a second hand. Thus the relevant counts are the same.

Mike Mailway claims that $P(B|A) > P(B)$. To verify this we need to get $P(A \cap B)$ and thus we need to count how many ways we can deal two hands, namely $\binom{52}{5} \binom{47}{5}$ ways, and how many sets of two hands can be dealt with each hand containing exactly one pair. This turns out to be quite complicated and not so automatic. We have to organize the count carefully. There are $1,098,240$ ways to fix the first hand. For the sake of argument, assume that the first hand consists of $\{K, K, Q, J, 10\}$ (the suits don’t matter).

We consider first all second hands which have a pair of same face value, i.e., kings. Since there is only one pair left, there is only one way to grab those kings for the second hand. Next, we have to decide what the other three card values should be. They could be again $\{Q, J, 10\}$ for which we have $3^3 = 27$ possibilities. We could have two distinct faces from $\{Q, J, 10\}$ and one from $\{9, 8, \ldots , A\}$. Of these we could have $\binom{3}{2} 3^2 \binom{9}{1} = 972$. We could also have one from $\{Q, J, 10\}$ and two different faces from $\{9, 8, \ldots , 2, A\}$, for a count of $\binom{3}{1} 3 \binom{9}{2} 4^2 = 5,184$. Finally, the remaining three could be three distinct faces from $\{9, 8, \ldots , 2, A\}$, for $\binom{3}{3} 4^3 = 5,376$ ways. Thus we have

$$27 + 972 + 5,184 + 5,376 = 11,559$$

second hands with a king pair only.
Now let us count the number of second hands with a pair face chosen from \( \{Q, J, 10\} \). There are \( 3 \binom{3}{2} = 9 \) such pairs that can be had. For the sake of argument we assume that a pair of queens was chosen. Next we fill the remaining three slots. We count separately the possibilities including a king and not including a king. Including a king has 2 choices and the other two could be from \( \{J, 10\} \) for \( 3^2 \) choices (queens are no longer possible because of the pair of queens), or the other two could be one from \( \{J, 10\} \) and one from \( \{9, 8, \ldots, A\} \) for \( 2 \cdot 3 \cdot 9 \cdot 4 = 216 \) choices, or both taken from \( \{9, 8, \ldots, A\} \) for \( \binom{9}{2} 4^2 = 576 \) choices. Thus we have

\[
9 \cdot 2 \cdot (9 + 216 + 576) = 14,418
\]

ways of getting a second hand with a pair with face value chosen from \( \{Q, J, 10\} \) and a king.

Now count the number of second hands with a pair face value chosen from \( \{Q, J, 10\} \) (say two queens again) but without a king. The remaining three cards could all be chosen with two distinct faces from \( \{J, 10\} \) and one from \( \{9, 8, \ldots, A\} \) for \( 3^2 \binom{9}{1} 4 = 324 \) ways, or one could be from \( \{J, 10\} \) and two distinct faces from \( \{9, 8, \ldots, A\} \) for \( \binom{9}{1} \binom{3}{2} 4^2 = 3,456 \) ways, or all three distinct faces could come from \( \{9, 8, \ldots, A\} \) for \( \binom{9}{3} 4^3 = 5,376 \) ways. Thus we would have

\[
9 \cdot (324 + 3,456 + 5,376) = 82,404
\]

second hands with a single pair with face from \( \{Q, J, 10\} \) and no king. Adding this to the above 14,418 we have 14,418 + 82,404 = 96,822 second hands with single pair with face from \( \{Q, J, 10\} \).

Finally, let the face value of the single pair for the second hand be chosen from \( \{9, 8, \ldots, A\} \). There are \( 9 \cdot \binom{4}{2} = 54 \) ways of doing that. Assume it is a pair of 9’s. The remaining three cards could be chosen as one from the two kings and two from \( \{Q, J, 10\} \) in \( 2 \cdot \binom{3}{2} \cdot 3^2 = 54 \) ways, or as a king, one from \( \{Q, J, 10\} \) and one from \( \{8, 7, \ldots, A\} \) for \( 2 \cdot \binom{3}{1} \cdot 3 \cdot \binom{8}{2} 4 = 576 \) ways, or as a king and two distinct faces from \( \{8, 7, \ldots, A\} \) for \( 2 \cdot \binom{8}{2} 4^2 = 896 \) ways, or as three distinct faces from \( \{Q, J, 10\} \) for \( 3^3 = 27 \) ways, or as two distinct faces from \( \{Q, J, 10\} \) and one from \( \{8, 7, \ldots, A\} \) for \( \binom{3}{2} 3^2 \binom{8}{1} 4 = 864 \) ways, or as one from \( \{Q, J, 10\} \) and two distinct faces from \( \{8, 7, \ldots, A\} \) for \( \binom{3}{1} 3 \cdot \binom{8}{2} 4^2 = 4,032 \) ways, or finally as three distinct faces from \( \{8, 7, \ldots, A\} \) for \( \binom{8}{3} 4^3 = 3,584 \) ways. Adding all these up we find

\[
54 \cdot (54 + 576 + 896 + 27 + 864 + 4,032 + 3,584) = 541,782
\]

second hands with a single pair chosen from \( \{9, 8, \ldots, A\} \).

Totalling all this up we have

\[
11,559 + 96,822 + 541,782 = 650,163
\]

possible second hands with a single pair after a first hand with a single pair was dealt. Thus we have

\[
P(A \cap B) = \frac{1,098,240 \cdot 650,163}{\binom{52}{5} \binom{47}{5}}
\]

and

\[
P(B \mid A) = \frac{P(A \cap B)}{P(A)} = \frac{650,163}{\binom{47}{5}} \approx \frac{650,163}{1,533,939} = .423852.
\]

Hence Mike Mailway was right, but by the slightest margin. This would hardly be noticed in thousands of poker games.