

Centrality

567 Statistical analysis of social networks

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Centrality

A common goal in SNA is to identify the “central” nodes of a network.

What does “central” mean?

- active?
- important?
- non-redundant?

Koschutski et al. (2005) attempted a classification of centrality measures

- Reach: ability of ego to reach other vertices
- Flow: quantity/weight of walks passing through ego
- Vitality: effect of removing ego from the network
- Feedback: a recursive function of alter centralities

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Common centrality measures

We will define and compare four centrality measures:

- **degree centrality (based on degree)**
- closeness centrality (based on average distances)
- betweenness centrality (based on geodesics)
- eigenvector centrality (recursive: similar to page rank methods)

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Standardized centrality measures

Node-level indices

Let c_1, \dots, c_n be node-level centrality measures:

$c_i =$ centrality of node i by some metric

It is often useful to standardize the c_i 's by their maximum possible value:

$$\tilde{c}_i = c_i / c_{\max}$$

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Network centralization

Network-level indices

How centralized is the network?

To what extent is there a small number of highly central nodes?

- Let $c^* = \max\{c_1, \dots, c_n\}$
- Let $S = \sum_i [c^* - c_i]$

Then

- $S = 0$ if all nodes are equally central;
- S is large if one node is most central.

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Network level centralization index

$$C = \frac{\sum_i [c^* - c_i]}{\max \sum_i [c^* - c_i]}$$

The “max” in the denominator is over all possible networks.

- $0 \leq C \leq 1$;
- $C = 0$ when all nodes have the same centrality;
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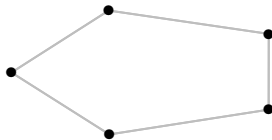
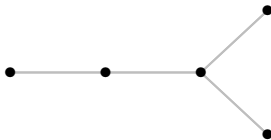
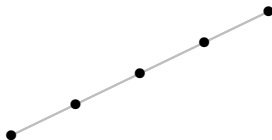
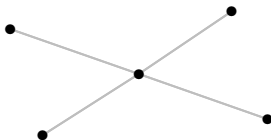
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Networks for comparison

We will compare the following graphs under different centrality measures:

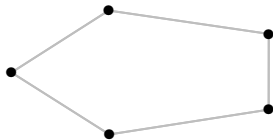
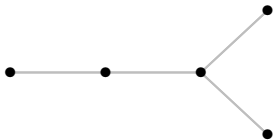
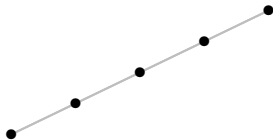
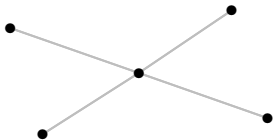


These are the star graph, line graph, y-graph, the circle graph.

Which do you feel is most "centralized"? Which the least?

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Degree centrality

Idea: A central actor is one with many connections.

This motivates the measure of **degree centrality**

- undirected degree centrality: $c_i^d = \sum_{j, j \neq i} y_{j,i}$
- outdegree centrality: $c_i^o = \sum_{j, j \neq i} y_{i,j}$
- indegree centrality: $c_i^i = \sum_{j, j \neq i} y_{j,i}$

The standardized degree centrality is

$$\check{c}_i^d = c_i^d / c_{\max}^d = c_i^d / (n - 1)$$

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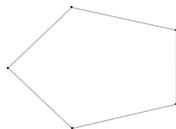
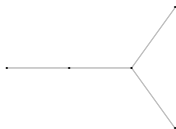
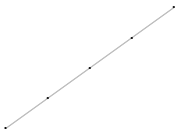
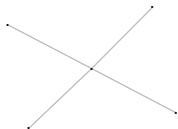
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Degree centrality



```
apply(Ys, 1, sum, na.rm=TRUE)
```

```
## 1 2 3 4 5
```

```
## 1 1 4 1 1
```

```
apply(Yl, 1, sum, na.rm=TRUE)
```

```
## 1 2 3 4 5
```

```
## 1 2 2 2 1
```

```
apply(Yy, 1, sum, na.rm=TRUE)
```

```
## 1 2 3 4 5
```

```
## 1 2 3 1 1
```

```
apply(Yc, 1, sum, na.rm=TRUE)
```

```
## 1 2 3 4 5
```

```
## 2 2 2 2 2
```

Degree centralization

c_i^d : actor centrality

c^{d*} : maximum actor centrality observed in the network

$\sum_i [c^{d*} - c_i^d]$: sum of differences between most central actor and others

Centralization

$$C^d = \frac{\sum_i [c^{d*} - c_i^d]}{\max_Y \sum_i [c^{d*} - c_i^d]}$$

What is the maximum numerator that could be attained by an n -node graph?

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Degree centralization

The maximum occurs when

- one node has the largest possible degree ($c^{d*} = n - 1$),
- the others have the smallest possible degree $c_i^d = 1$.

This is the star graph.

$$\begin{aligned}\max_{\mathbf{Y}} \sum_i [c^{d*} - c_i^d] &= \sum_i [(n-1) - c_i^d] \\ &= 0 + (n-1-1) + \cdots + (n-1-1) \\ &= (n-1)(n-2)\end{aligned}$$

$$C^d(\mathbf{Y}) = \frac{\sum_i [c^{d*} - c_i^d]}{(n-1)(n-2)}$$

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Degree centralization

Exercise: Compute the degree centralization for the four $n = 5$ graphs:

- the star graph;
- the line graph;
- the y-graph;
- the circle graph.

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Cd<-function(Y)
{
  n<-nrow(Y)
  d<-apply(Y,1,sum,na.rm=TRUE)
  sum(max(d)-d)/( (n-1)*(n-2) )
}
```

```
Cd(Ys)
```

```
## [1] 1
```

```
Cd(Yy)
```

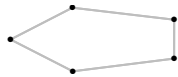
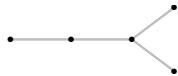
```
## [1] 0.5833333
```

```
Cd(Yl)
```

```
## [1] 0.1666667
```

```
Cd(Yc)
```

```
## [1] 0
```



Closeness centrality

Idea: A central node is one that is close, on average, to other nodes.

This motivates the idea of **closeness centrality**

- (geodesic) distance: $d_{i,j}$ is the minimal path length from i to j ;
- closeness centrality: $c_i^c = 1 / \sum_{j \neq i} d_{i,j} = 1 / [(n-1)\bar{d}_i]$;
- limitation: not useful for disconnected graphs.

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Idea: A central node is one that is close, on average, to other nodes.

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$$c_i^c = 1/[(n-1)\bar{d}_i]$$

Recall,

$$d_a < d_b \Rightarrow \frac{1}{d_a} > \frac{1}{d_b}$$

and so a node i would be “maximally close” if $d_{i,j} = 1$ for all $j \neq i$.

$$c_{\max}^d = \frac{1}{n-1}$$

The standardized closeness centrality is therefore

$$\begin{aligned}\tilde{c}_i^c &= c_i^c / c_{\max}^d \\ &= (n-1)c_i^c = 1/\bar{d}_i.\end{aligned}$$

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Closeness centralization

c_i^c : actor centrality

c^{c*} : maximum actor centrality observed in the network

$\sum_i [c^{c*} - c_i^c]$: sum of differences between most central actor and others

Centralization

$$C^c = \frac{\sum_i [c^{c*} - c_i^c]}{\max_Y \sum_i [c^{c*} - c_i^c]}$$

What is the maximum numerator that could be attained by an n -node graph?

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The maximum occurs when

- one node has the largest possible closeness ($\bar{d}^* = 1, c^{c^*} = 1/(n - 1)$),
- the others have the smallest possible closeness, given that $c^{c^*} = 1/(n - 1)$.

(Freeman, 1979)

For what graph are these conditions satisfied?

- For $c^{c^*} = 1/(n - 1)$, one node must be connected to all others.
- To then maximize centralization, the centrality of the other nodes must be minimized.

This occurs when none of the non-central nodes are tied to each other, i.e. the star graph.

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Closeness centralization

For a non-central node in the star graph,

$$\begin{aligned}\bar{d}_i &= \frac{1 + 2 + \dots + 2}{n - 1} \\ &= \frac{2(n - 2) + 1}{n - 1} \\ &= \frac{2n - 3}{n - 1} \\ c_i^c &= 1/[(n - 1)\bar{d}_i] = \frac{1}{2n - 3}.\end{aligned}$$

Therefore, for the star graph

$$\begin{aligned}\sum_i [c^{c*} - c_i^c] &= 0 + \left(\frac{1}{n - 1} - \frac{1}{2n - 3}\right) + \dots + \left(\frac{1}{n - 1} - \frac{1}{2n - 3}\right) \\ &= (n - 1) \times \left(\frac{1}{n - 1} - \frac{1}{2n - 3}\right) \\ &= (n - 1) \times \frac{n - 2}{(2n - 3)(n - 1)} \\ &= \frac{n - 2}{2n - 3}\end{aligned}$$

Closeness centralization

To review, the maximum of $\sum_i [c^{c^*} - c_i^c]$ occurs for the star graph, for which

$$\sum_i [c^{c^*} - c_i^c] = \frac{n-2}{2n-3}$$

Therefore, the centralization of any graph \mathbf{Y} is

$$\begin{aligned} C^c(\mathbf{Y}) &= \frac{\sum_i [c^{c^*} - c_i^c]}{\max_{\mathbf{Y}} \sum_i [c^{c^*} - c_i^c]} \\ &= \frac{\sum_i [c^{c^*} - c_i^c]}{(n-2)/(2n-3)} \end{aligned}$$

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Closeness centralization

Exercise: Compute the closeness centralization for the four $n = 5$ graphs:

- the star graph;
- the line graph;
- the y-graph;
- the circle graph.

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Closeness centralization

```
Cc<-function(Y)
{
n<-nrow(Y)
D<-netdist(Y)
c<-1/apply(D,1,sum,na.rm=TRUE)

sum(max(c)-c)/( (n-2)/(2*n-3) )
}

Cc(Ys)

## [1] 1

Cc(Yy)

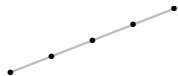
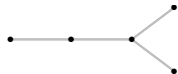
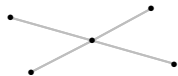
## [1] 0.6351852

Cc(Yl)

## [1] 0.4222222

Cc(Yc)

## [1] 0
```



Betweenness centrality

Idea: A central actor is one that acts as a bridge, broker or gatekeeper.

- Interaction between unlinked nodes goes through the shortest path (geodesic);
- A “central” node is one that lies on many geodesics.

This motivates the idea of **betweenness centrality**

- $g_{j,k}$ = number of geodesics between nodes j and k ;
- $g_{j,k}(i)$ = number of geodesics between nodes j and k going through i ;
- $c_i^b = \sum_{j < k} g_{j,k}(i) / g_{j,k}$

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Interpretation: $g_{j,k}(i)/g_{j,k}$ is the probability that a “message” from j to k goes through i .

- j and k have $g_{j,k}$ routes of communication;
- i is on $g_{j,k}(i)$ of these routes;
- a randomly selected route contains i with probability $g_{j,k}(i)/g_{j,k}$.

Note: WF p.191

“(betweenness centrality) can be computed even if the graph is not connected”
(WF)

- Careful: If j and k are not reachable, what is $g_{j,k}(i)/g_{j,k}$?
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- $0 \leq c_i^b$, with equality when i lies on no geodesics (draw a picture)
- $c_i^b \leq \binom{n-1}{2} = \frac{(n-1)(n-2)}{2}$, with equality when i lies on all geodesics.

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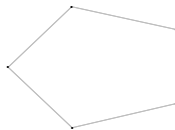
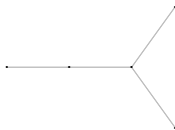
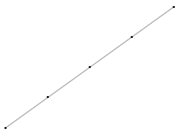
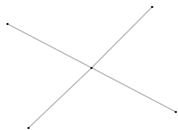
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Betweenness centrality



Exercise: Compute the betweenness centrality for each node in each graph.

```
betweenness(Ys,gmode="graph")
```

```
## [1] 0 0 6 0 0
```

```
betweenness(Yl,gmode="graph")
```

```
## [1] 0 3 4 3 0
```

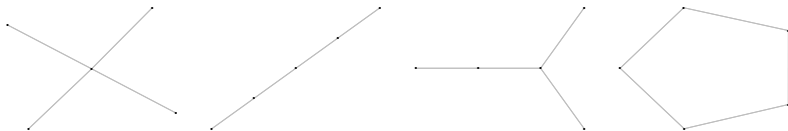
```
betweenness(Yy,gmode="graph")
```

```
## [1] 0 3 5 0 0
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```
betweenness(Yc,gmode="graph")
```

```
## [1] 1 1 1 1 1
```

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c^{b*} : maximum actor centrality observed in the network

$\sum_i [c^{b*} - c_i^b]$: sum of differences between most central actor and others

Centralization

$$C^b = \frac{\sum_i [c^{b*} - c_i^b]}{\max_Y \sum_i [c^{b*} - c_i^b]}$$

What is the maximum numerator that could be attained by an n -node graph?

Betweenness centralization

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Betweenness centralization

The maximum occurs when

- one node has the largest possible betweenness ($c^{b*} = \binom{n-1}{2}$),
- the others have the smallest possible betweenness ($c_i^d = 0$).

Again, this is the star graph.

$$\begin{aligned}\max_{\mathbf{Y}} \sum_i [c^{b*} - c_i^b] &= \sum_i \left[\binom{n-1}{2} - c_i^b \right] \\ &= 0 + \left(\binom{n-1}{2} - 0 \right) + \dots + \left(\binom{n-1}{2} - 0 \right) \\ &= (n-1) \binom{n-1}{2}\end{aligned}$$

$\binom{n-1}{2} = (n-1)(n-2)/2$, so

$$\begin{aligned}C^b(\mathbf{Y}) &= \frac{\sum_i [c^{b*} - c_i^b]}{(n-1) \binom{n-1}{2}} \\ &= 2 \frac{\sum_i [c^{b*} - c_i^b]}{(n-1)^2 (n-2)}\end{aligned}$$

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Betweenness centralization

```
Cb<-function(Y)
{
  require(sna)
  n<-nrow(Y)
  b<-betweenness(Y,gmode="graph")
  2*sum(max(b)-b)/( (n-1)^2 * (n-2) )
}
```

```
Cb(Ys)
```

```
## [1] 1
```

```
Cb(Yy)
```

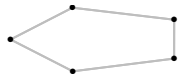
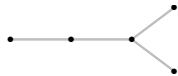
```
## [1] 0.7083333
```

```
Cb(Yl)
```

```
## [1] 0.4166667
```

```
Cb(Yc)
```

```
## [1] 0
```



Eigenvector centrality

Idea: A central actor is connected to other central actors.

This definition is recursive:

Eigenvector centrality: The centrality of each vertex is proportional to the sum of the centralities of its neighbors

- Formula: $c_i^e = \frac{1}{\lambda} \sum_{j \neq i} y_{ij} c_j^e$
- Central vertices are those with many central neighbors
- A variant of eigenvector centrality is used by Google to rank Web pages

Google Describing PageRank: *PageRank relies on the uniquely democratic nature of the web by using its vast link structure as an indicator of an individual page's value. In essence, Google interprets a link from page A to page B as a vote, by page A, for page B. But, Google looks at more than the sheer volume of votes, or links a page receives; it also analyzes the page that casts the vote. Votes cast by pages that are themselves "important" weigh more heavily and help to make other pages "important."*

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$$c_i^e = \frac{1}{\lambda} \sum_{j:j \neq i} y_{i,j} c_j^e$$

Using matrix algebra, such a vector of centralities satisfies

$$\mathbf{Y}\mathbf{c}^e = \lambda\mathbf{c}^e,$$

where the missing diagonal of \mathbf{Y} has been replaced with zeros.

A vector \mathbf{c}^e satisfying the above equation is an **eigenvector** of \mathbf{Y} .

There are generally multiple eigenvectors. The centrality is taken to be the one corresponding to the largest value of λ .

- this corresponds with the best rank-1 approximation to \mathbf{Y} ;
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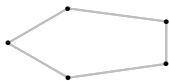
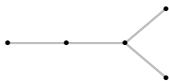
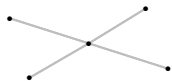
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Eigenvector centrality



```
evecc<-function(Y)
{
  diag(Y)<-0
  tmp<-eigen(Y)$vec[,1] ; tmp<-tmp*sign(tmp[1])
  tmp
}

evecc(Ys)

## [1] 0.3535534 0.3535534 0.7071068 0.3535534 0.3535534

evecc(Yl)

## [1] 0.2886751 0.5000000 0.5773503 0.5000000 0.2886751

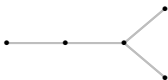
evecc(Yy)

## [1] 0.2705981 0.5000000 0.6532815 0.3535534 0.3535534

evecc(Yc)

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Eigenvector centralization

```
Ce<-function(Y)
{
  n<-nrow(Y)
  e<-evecc(Y)
  Y.sgn<-matrix(0,n,n) ; Y.sgn[1,-1]<-1 ; Y.sgn<-Y.sgn+t(Y.sgn)
  e.sgn<-evecc(Y.sgn)
  sum(max(e)-e)/ sum(max(e.sgn)-e.sgn)
}
```

```
Ce(Ys)
```

```
## [1] 1
```

```
Ce(Yy)
```

```
## [1] 0.802864
```

```
Ce(Yl)
```

```
## [1] 0.5176381
```

```
Ce(Yc)
```

```
## [1] 9.420555e-16
```

Empirical study: Comparing centralization of different networks

Comparison of centralization metrics across four networks:

- `butland_ppi`: binding interactions among 716 yeast proteins
- `addhealth9`: friendships among 136 boys
- `tribes`: positive relations among 12 NZ tribes

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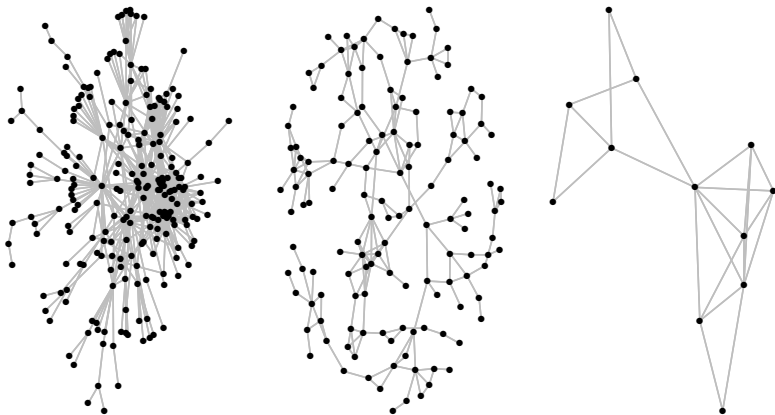
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	degree	closeness	betweenness	eigenvector
ppi	0.13	0.26	0.31	0.35
addhealth	0.04	0.14	0.42	0.61
tribes	0.35	0.5	0.51	0.47

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Comments:

- The protein network looks visually centralized, but
 - most centralization is local;
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- The friendship network has small degree centrality (why?).
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