Inference for stochastic processes in environmental science

III: Nonstationary covariance structure

Peter Guttorp
NRCSE

Outline

The deformation approach
How general is it?
A Bayesian implementation
Application to point processes
Global covariance structures
General setup

\[ Z(x,t) = \mu(x,t) + \nu(x)^{1/2}E(x,t) + \epsilon(x,t) \]

\[ \text{trend + smooth + error} \]

We shall assume that \( \mu \) is known or constant.

t = 1,...,T indexes temporal replications

E is L_2-continuous, mean 0, variance 1, independent of the error \( \epsilon \)

\( C(x,y) = \text{Cor}(E(x,t),E(y,t)) \)

\( D(x,y) = \text{Var}(E(x,t)-E(y,t)) \) (dispersion)

\[ \text{Cov}(Z(x,t),Z(y,t)) = \begin{cases} \sqrt{\nu(x)\nu(y)}C(x,y) & x \neq y \\ \nu(x) + \sigma_\epsilon^2 & x = y \end{cases} \]

Geometric anisotropy

Recall that if \( C(x,y) = C(||x - y||) \) we have an isotropic covariance (circular isocorrelation curves).

If \( C(x,y) = C(||Ax - Ay||) \) for a linear transformation \( A \), we have geometric anisotropy (elliptical isocorrelation curves).

General nonstationary correlation structures are typically locally geometrically anisotropic.
The deformation idea

In the geometric anisotropic case, write
\[ C(x, y) = C(||f(x) - f(y)||) \]
where \( f(x) = Ax \). This suggests using a general nonlinear transformation \( f: R^2 \to R^d \). Usually \( d = 2 \) or 3.

We do not want \( f \) to fold.

Implementation

Consider observations at sites \( x_1, \ldots, x_n \). Let \( \hat{C}_{ij} \) be the empirical covariance between sites \( x_i \) and \( x_j \). Minimize
\[ (\theta, f) \mapsto \sum_{i,j} w_{ij} (\hat{C}_{ij} - C(f(x_i), f(x_j); \theta))^2 + \lambda J(f) \]
where \( J(f) \) is a penalty for non-smooth transformations, such as the bending energy
\[ J(f) = \iint \left[ \left( \frac{\partial^2 f}{\partial x^2} \right)^2 + 2 \left( \frac{\partial^2 f}{\partial x \partial y} \right)^2 + \left( \frac{\partial^2 f}{\partial y^2} \right)^2 \right] \, dx \, dy \]
SARMAP

An ozone monitoring exercise in California, summer of 1990, collected data on some 130 sites.

Transformation

This is for hr. 16 in the afternoon
**Identifiability**

Perrin and Meiring (1999): Let
\[ D(x,y) = \gamma(||f(x) - f(y)||) \ (x,y) \in \mathbb{R}^n \times \mathbb{R}^n \]
If (1) \( f \) and \( f^{-1} \) are differentiable in \( \mathbb{R}^n \)
(2) \( \gamma(u) \) is differentiable for \( u > 0 \)
then \((f,\gamma)\) is unique, up to a scaling for \( \gamma \)
and a homothetic transformation for \( f \)
(rotation, scaling, reflection)

**Richness**

Perrin & Senoussi (2000): Let \( f \) and \( f^{-1} \) be differentiable, and let \( r(x,y) \) be continuously differentiable. Then
\[ r(x,y) = \rho(||f(x) - f(y)||) \ (\text{stationary}) \iff \]
\[ (*) \frac{\partial}{\partial x} r(x,y) J^{-1}(x) + \frac{\partial}{\partial y} r(x,y) J^{-1}(y) = 0, x \neq y \]
Let \( f(0) = 0, c_i(0) \) \( i^{th} \) column of \( J_i^{-1}(0) \).
Then \( r(x,y) = \rho(||f(x) - f(y)||) \ (\text{isotropy}) \iff \)
\[ (*) \text{ and } f_i(y) \frac{\partial}{\partial x} r(0,y) c_i(0) = f_j(y) \frac{\partial}{\partial x} r(0,y) c_j(0) \]
Brownian sheet can be transformed to stationarity, but not to isotropy.
Estimating variability

Resample time slices with replacement from the original data (to maintain spatial structure). Reestimate deformation based on each bootstrap sample. Kriging estimates can be made based on each of the bootstrap estimates, to get a better sense of the variability.

French rainfall data

Variability of estimated dispersions with site 45

50 bootstrap samples.

Determination of the smoothing parameter $\lambda$

Cross-validation:
Leave out sampling station $i$, estimate $(\theta, f)$ from remaining $n-1$ stations. Now minimize the prediction error for site $i$, summed over $i$.
Together with bootstrap estimate of variability, very computer intensive.
A Bayesian approach

For a Gaussian process with constant mean $m$, integrated out using a flat prior, the likelihood for the covariance matrix $\Sigma$ has the Wishart form

$$L(\Sigma \mid \{z_i\}) = (2\pi \det \Sigma)^{-1/2} \exp\left(-\frac{T}{2} \text{tr} \Sigma^{-1} S\right)$$

where $S$ is the sample covariance.

The prior on the transformation $f$ is taken proportional to the bending energy, scaled by a smoothing parameter $\tau$.

Estimation by MCMC.

French precipitation data
Uncertainty in deformation

Model refitted using 24 of the 36 sites.

Point process deformations

Markov point processes are determined by a neighborhood system and a density \( f \) with respect to a Poisson process measure. By Hammersley-Clifford, the density can be written using a clique interaction function \( \phi \):

\[
f(x) = \prod_{y \subseteq x} \phi(y)
\]

A deformation of a Markov point process using a bijection \( h \) yields a new Markov point process.
Point process deformations, cont.

The new process has neighborhood system given by $y_1 \sim y_2$ iff $h^{-1}(y_1) \sim h^{-1}(y_2)$

The density of the transformed process has clique interaction function $\psi$, given by

$$\psi(z) = \begin{cases} 
\phi(\emptyset) & \text{if } n(z) = 0 \\
\phi(h^{-1}(\eta))Jh^{-1}(\eta) & \text{if } n(z) = 1, z = \{\eta\} \\
\phi(h^{-1}(\eta)) & \text{otherwise}
\end{cases}$$

where $Jh$ is the Jacobian of $h$, given by

$$\int_x g(x)Jh(x)\lambda(dx) = \int_y g(h^{-1}(y))\lambda(dy)$$

and density

$$f(y) = \prod_{\eta \in y} Jh^{-1}(\eta) \prod_{z \subset h^{-1}(y)} \phi(z)$$

An example

Consider mapping the unit sphere in $\mathbb{R}^3$ onto itself. A class of invertible mappings is given by

$$h_0(x_1, x_2, x_3) = \left( \frac{\sqrt{1 - r_0(x_3)^2} - x_1}{\sqrt{1 - x_2^2}}, \frac{\sqrt{1 - r_0(x_3)^2} - x_2}{\sqrt{1 - x_3^2}}, r_0(x_3) \right)$$

where $r_0(x) = \frac{1}{\theta} \log \left( \frac{u - 1}{2} \left( e^0 - e^{-\theta} \right) + e^0 \right)$, $-1 \leq u \leq 1$

Strauss process
Global processes

Problems such as global warming require modeling of processes that take place on the globe (an oriented sphere). Optimal prediction of quantities such as global mean temperature need models for global covariances.

Note: spherical covariances can take values in \([-1,1]\)–not just imbedded in \(\mathbb{R}^3\). Also, stationarity and isotropy are identical concepts on the sphere.

Isotropic covariances on the sphere

Isotropic covariances on a sphere are of the form

\[
C(p,q) = \sum_{i=0}^{\infty} a_i P_i(\cos \gamma_{pq})
\]

where \(p\) and \(q\) are directions, \(\gamma_{pq}\) the angle between them, and \(P_i\) the Legendre polynomials.

Example: \(a_i=(2i+1)i\)

\[
C(p,q) = \frac{1 - \rho^2}{1 - 2\rho \cos \gamma_{pq} + \rho^2} - 1
\]
A class of global transformations

Iteration between simple parametric deformation of latitude (with parameters changing with longitude) and similar deformations of longitude (changing smoothly with latitude).

Three iterations
Global temperature

Global Historical Climatology Network
7280 stations with at least 10 years of data. Subset with 839 stations with data 1950-1991 selected.

Isotropic correlations
References

**Spatial deformation models**


**Point processes**


**Global processes**