Conditional Modelling of Extreme Wind Gusts by Bivariate Brown-Resnick Processes

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joint work with
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Recall the talks by D. Cooley and M. Ribatet

Extreme value theory

- **Generalized Extreme Value (GEV) Distribution**
  - limit law of i.i.d. maxima
  - annual maxima

- **Generalized Pareto Distribution**
  - exceedances over high thresholds
  - tail equivalent to GEV

- **Max-stable processes**
  - spatial concept of extremes
  - statistical inference
Outline

1. Construction of max-stable processes
2. Tail correlation functions
3. Weather forecasting
4. Data
5. Marginal Model
6. Dependence Model
7. Application to Data
Spectral Representation (de Haan, 1984)

- \( K \subset \mathbb{R}^d \) compact
- \( \mathcal{H} \): space of spectral functions \( K \rightarrow [0, \infty) \) with measure \( H \)
- \( \Pi = \sum \delta(u_i, v_i) \): Poisson point process on \((0, \infty) \times \mathcal{H}\) with intensity \( u^{-2} du \cdot H(df) \)

\[
X(t) = \max_{i \in \mathcal{N}} U_i V_i(t), \quad t \in K
\]
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Moving Maxima (e.g. Smith Process, 1990)

- \( \sum_{i \in \mathcal{N}} \delta(u_i, s_i) \): Poisson point process on \((0, \infty) \times \mathbb{R}^d\) with intensity \(u^{-2} du \times ds\)
- \(F\): deterministic “shape function”

\[
X(t) = \max_{i \in \mathcal{N}} (U_i \cdot F(t - S_i))
\]

\(\sim\) spectral functions are shifted shape functions \(F(\cdot - S_i)\)
What do these pictures have in common?
Tail correlation function (TCF)

\[ X = \{X(t)\}_{t \in \mathbb{R}^d}: \text{a stationary stochastic process} \]

\[ \chi(t) := \lim_{x \to x^*} \mathbb{P}(X_t > x \mid X_0 > x), \quad t \in \mathbb{R}^d. \]

(provided limits exist; \( x^* \) = upper endpoint)

Comments

- correlation function for tail dependence
- invariant under continuous isotonic marginal transformations
- different names in the literature:
  - (upper) tail dependence coefficient [Beirlant et al. '04, Davis/Mikosch '09, Falk '05]
  - \( \chi \)-measure [Beirlant et al. '04, Coles et al. '99]
  - extremal coefficient function [Fasen et al. '10]
  - …

- estimable by \( F \)-madogram (Cooley, Naveau & Poncet, 2006)
Properties

\( \chi(t) = \lim_{x \to x^*} \mathbb{P}(X_t > x | X_o > x), \quad t \in \mathbb{R}^d \)

- \( \chi \) is positive semidefinite (a non-negative correlation function):
  \[
  \sum_{i=1}^{n} \sum_{j=1}^{n} a_i \chi(t_i - t_j) a_j \geq 0, \quad \forall (t_1, \ldots, t_n) \in (\mathbb{R}^d)^n, \quad \forall (a_1, \ldots, a_n) \in \mathbb{R}^n.
  \]

- \( \chi \) satisfies further inequalities:
  \[
  \chi_{ij} \geq 0 \\
  \chi_{ij} + \chi_{ik} - \chi_{jk} \leq 1 \\
  (\chi_{ij} + \chi_{ik} + \chi_{il}) - (\chi_{jk} + \chi_{jl} + \chi_{kl}) \leq 1
  \]
  \( \forall (t_i, t_j, t_k, t_l) \in (\mathbb{R}^d)^4 \)
  with \( \chi_{ij} = \chi(t_i - t_j) \).

[Davis/Mikosch '09, S./Tawn '03, Strokorb '13]
Example: Parametric families

**Powered Exponential**

\[ \rho_{\alpha}(r) = \exp(-r^{\alpha}) \]

**Whittle-Matérn**

\[ \rho_{\alpha}(r) = \frac{2^{1-\alpha}}{\Gamma(\alpha)} r^{\alpha} K_{\alpha}(r) \]

**Cauchy**

\[ \rho_{\alpha,\beta}(r) = (1 + r^{\alpha})^{-\beta} \]

CF for \( \alpha \in (0, 2] \)

CF for \( \alpha \in (0, \infty) \)

CF for \( \alpha \in (0, 2] \) (for all \( \beta > 0 \))
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**Cauchy**
\[ \rho_{\alpha,\beta}(r) = (1 + r^\alpha)^{-\beta} \]

**CF for** \( \alpha \in (0, 2] \)

**CF for** \( \alpha \in (0, \infty) \)

**CF for** \( \alpha \in (0, 2] \) (for all \( \beta > 0 \))

**TCF for** \( \alpha \in (0, 1] \)

**TCF for** \( \alpha \in (0, 0.5] \)

**TCF for** \( \alpha \in (0, 1] \) (for all \( \beta > 0 \))
Construction

- Moving maxima process:

\[ X(t) = \max_{(u,s) \in \Pi} u F(t - s) \]

\( \Pi \) a Poisson point process with intensity \( u^{-2} \, du \, ds \)

- Deterministic shape function \( F \) in \( \mathbb{R}^3 \),

\[ F(t) = \frac{1 + 4 \|t\|}{\pi^{3/2} \|2t\|^{5/2}} e^{-2\|t\|} \]

- Background picture is the 2-dimensional cross-section of a 3-dimensional realisation of \( \log(X) \)
Properties

- tail correlation function equals

\[ \chi(h) = \lim_{x \to x^*} \mathbb{P}(X(h) > x \mid X(0) > x) = \text{erfc}(\sqrt{\|h\|}) \]

where \( \text{erfc}(x) = \frac{2}{\sqrt{\pi}} \int_x^\infty e^{-y^2} \, dy \)
**Properties**

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I.e., identical to the (classical) Brown-Resnick process.
Background picture, part II

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where \( \text{erfc}(x) = \frac{2}{\sqrt{\pi}} \int_x^\infty e^{-y^2} \, dy \)

I.e., identical to the (classical) Brown-Resnick process.

- discontinuous everywhere
So, these pictures have the tail correlation function in common!
Numerical weather forecast

System of six partial differential equations
- equations include conservation of momentum, mass, energy and entropy, and equation of state
- two velocity components, density, pressure, temperature, humidity

Deterministic forecasts of future states of the atmosphere
- discretization
- run forward in time

Initial conditions
- data assimilation systems
- describing current state of the atmosphere on a 3d grid
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Tim Palmer (2000):
Although forecasters have traditionally viewed weather prediction as deterministic, a culture change towards probabilistic forecasting is in progress.
Postprocessing

NWP ensembles are subject to model biases and typically they show a lack of calibration

Univariate postprocessing

- regression based approach (Gneiting et al., 2005) using normal distribution assumptions
- Bayesinan approach (Raftery et al., 2005)

Multivariate and spatial postprocessing

- empirical copula coupling (Schuhlen et al., 2012)
- score functions
Postprocessing of extremes

- ansatz with normal distribution should be replaced by GEV
- empirical copula coupling coupling might be replaced by a spatial model
- quality control essential
- spatial model, hence downscaling, might be worthwhile
Extreme Wind Gusts

- wind gusts are strongly varying in space
- high uncertainty in forecasts, particularly for extreme wind gusts
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Goal:

Model for the (observed) wind gusts $V_{\text{max}}^{\text{obs}}$ conditional on the forecast $V_{\text{max}}^{\text{pred}}$
Model for Extreme Wind Gusts

Goal:
Model for the (observed) wind gusts $V_{\text{obs}}^{\text{max}}$ conditional on the forecast $V_{\text{pred}}^{\text{max}}$

Two Modelling Steps:
1. model for marginal distributions (at single location) of $V_{\text{max}}^{\text{obs}}$ & $V_{\text{max}}^{\text{pred}}$
2. model for spatial dependence & dependence between observation and prediction
   $\sim$ bivariate stochastic process
The Data

**Observation data:**
for the maximal wind speed $V_{\text{obs}}^{\text{max}}$
- at 116 DWD stations in Northern Germany
- for 358 days (03/2011 – 02/2012)

**Forecast data:**
from COSMO-DE EPS on a grid with mesh size 2.8 km covering Germany
- 20 ensemble members
- for the maximal wind speed $V_{\text{pred}}^{\text{max}}$
The Data

**Observation data:**
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**Forecast data:**
from COSMO-DE EPS
- on a grid with mesh size 2.8 km covering Germany
- 20 ensemble members
1. for the maximal wind speed $V_{\text{pred max}}$
2. for the mean wind speed
Marginal Model for Wind Speed

(single) wind speed \( V(l, d) \) at location \( l \) and day \( d \):

\[
V(l, d) = d \sqrt{\text{Var}\{V(l, d)\}} V_0 + \mathbb{E}\{V(l, d)\}
\]

with \( V_0 \) following some standardized distribution

“weather parameters” \( \mathbb{E}\{V(l, d)\} \) and \( \text{Var}\{V(l, d)\} \):

- reflect the general weather situation
- contain seasonal effects
- are assumed to be “known” to the forecaster
Marginal Model for Wind Speed

(single) wind speed $V(l, d)$ at location $l$ and day $d$:

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“weather parameters” $\mathbb{E}\{V(l, d)\}$ and $\text{Var}\{V(l, d)\}$:
- reflect the general weather situation
- contain seasonal effects
- are assumed to be “known” to the forecaster

can be estimated from mean wind ensemble forecast
Marginal Model for Extreme Wind Gusts

(single) wind speed $V(l, d)$ at location $l$ and day $d$:

$$V(l, d) = d \sqrt{\text{Var}\{V(l, d)\}} V_0 + \mathbb{E}\{V(l, d)\}$$

with $V_0$ following some standardized distribution

observed single wind speed: 3-second average
observed maximal wind speed: highest 3-second average per day

$\sim$ GEV

maximal wind speed $V_{\text{max}}(l, d)$ at location $l$ and day $d$:

$$\mathbb{P}\left( \frac{V_{\text{max}}(l, d) - \mathbb{E}\{V(l, d)\}}{\sqrt{\text{Var}\{V(l, d)\}}} \leq x \right) \approx \exp \left( - \left( 1 + \frac{x - \mu}{\sigma} \right)^{-1/\xi} \right) = G_{\xi, \mu, \sigma}(x)$$
Marginal Model for Extreme Wind Gusts (cont’d)

Observations

\[
\frac{V_{\text{max}}^{\text{obs}}(l, d) - \mathbb{E}\{V(l, d)\}}{\sqrt{\text{Var}\{V(l, d)\}}} \sim \text{GEV}(\xi^{\text{obs}}, \mu^{\text{obs}}, \sigma^{\text{obs}})
\]

Forecast

\[
\frac{V_{\text{max}}^{\text{pred}}(l, d) - \mathbb{E}\{V(l, d)\}}{\sqrt{\text{Var}\{V(l, d)\}}} \sim \text{GEV}(\xi^{\text{pred}}, \mu^{\text{pred}}, \sigma^{\text{pred}})
\]

GEV parameters \((\xi, \mu, \sigma)\):

- \(\xi\) constant in space in time
- error model allows \(\mu, \sigma\) to vary spatially
- estimated via maximum likelihood
GEV Parameters

Estimates for $\mu^{\text{obs}}$

Estimates for $\mu^{\text{pred}}$

Estimates for $\sigma^{\text{obs}}$

Estimates for $\sigma^{\text{pred}}$
necessity to model spatial prediction and spatial observation together
Intrinsically stationary processes

- **univariate case:**
  - $Y_s(t) = W^{(1)}(t + s) - W^{(1)}(t)$ is (weakly) stationary
  - variogramm $\gamma$,
    \[ \gamma(t, s) = \text{Var}(W^{(1)}(t) - W^{(1)}(s)) \]
    depends only on the distance vector $t - s$

- **multivariate set up:**
  - different approaches
  - we need that the pseudo-variogram $\gamma(s, t) = (\gamma_{ij}(s, t))_{1 \leq i, j \leq 2}$ with
    \[ \gamma_{ij}(s, t) = \text{Var}(W^{(i)}(s) - W^{(j)}(t)) \]
    only depends on $s - t$. 
Point Process Construction (Brown & Resnick 1977)

- \( \{U_k, k \in \mathcal{N}\} \): PPP on \( \mathbb{R} \) with intensity measure \( e^{-u} \, du \) (magnitude)
Point Process Construction (Brown & Resnick 1977)

- \( \{ U_k, k \in \mathcal{N} \} \): PPP on \( \mathbb{R} \) with intensity measure \( e^{-u} \, du \) (magnitude)
- \( W(\cdot) \): standard Brownian motion (BM)
  \( (W_k(\cdot) - |\cdot|/2) \sim_{i.i.d} (W(\cdot) - |\cdot|/2) \) (spatial course)
Point Process Construction (Brown & Resnick 1977)

\[ X(t) = \max_{k \in \mathcal{N}} \left( U_k + W_k(t) - \frac{|t|}{2} \right), \quad t \in \mathbb{R} \]
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\( X \) is max-stable
\[ X(t) = \max_{k \in \mathbb{N}} \left( U_k + W_k(t) - \frac{|t|}{2} \right) \]

\( X \) is max-stable and stationary
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$X$ is max-stable and stationary
Generalization

(cf. Kabluchko et al. 2009)

- $\{U_k\}_{k \in \mathbb{N}}$: Poisson point process with intensity $e^{-u} \, du$
- $W(\cdot)$: centered Gaussian process on $\mathbb{R}^d$ s.t.

  \[
  \gamma(s, t) = \text{Var}(W(s) - W(t))
  \]

  depends on $s - t \in \mathbb{R}^d$ only

- $\{W_k(\cdot)\}_{k \in \mathbb{N}}$: independent copies of $W$
Generalization
(cf. Kabluchko et al. 2009)

- $\{U_k\}_{k\in\mathbb{N}}$: Poisson point process with intensity $e^{-u} \, du$
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  variogram
  
  $\gamma(s, t) = \text{Var}(W(s) - W(t))$

  depends on $s - t \in \mathbb{R}^d$ only

- $\{W_k(\cdot)\}_{k\in\mathbb{N}}$: independent copies of $W$

\[ X(t) = \max_{k\in\mathbb{N}} (U_k + W_k(t) - \text{Var}(W_k(t))/2), \quad t \in \mathbb{R}^d, \]

$X$ is called Brown-Resnick process associated to the variogram $\gamma$. 
Multivariate Generalization
(cf. Stucki & Molchanov, 2013, Oesting et al., 2013)

- $\{U_k\}_{k \in \mathbb{N}}$: Poisson point process with intensity $e^{-u} du$
- $W(\cdot)$: centered Gaussian process s.t.

\[
\text{variogram} \\
\gamma(s, t) = \text{Var}(W(s) - W(t))
\]

depends on $s - t \in \mathbb{R}^d$ only

- $\{W_k(\cdot)\}_{k \in \mathbb{N}}$: independent copies of $W$

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Then,
- $X$ is max-stable and stationary
- law of $X$ depends on $\gamma$ only
Multivariate Generalization
(cf. Stucki & Molchanov, 2013, Oesting et al., 2013)

- \( \{U_k\}_{k \in \mathcal{N}} \): Poisson point process with intensity \( e^{-u} \, du \)
- \( W(\cdot) = (W^{(1)}(\cdot), W^{(2)}(\cdot)) \): centered Gaussian process s.t. pseudo-variogram
  \[ \gamma(s, t) = \text{Var}(W^{(i)}(s) - W^{(j)}(t)) \text{ for } i, j = 1, 2 \]
  depends on \( s - t \in \mathbb{R}^d \) only
- \( \{W_k(\cdot)\}_{k \in \mathcal{N}} \): independent copies of \( W \)

\[ X^{(i)}(t) = \max_{k \in \mathcal{N}} \left( U_k + W^{(i)}_k(t) - \text{Var}(W^{(i)}_k(t))/2 \right), \quad t \in \mathbb{R}^d, \ i = 1, 2, \]

Then,
- \( X \) is max-stable and stationary (as bivariate process)
- law of \( X \) depends on \( \gamma \) only
What does a pseudo-variogram look like?

\[
\gamma(s, t) = \text{Var}(W^{(i)}(s) - W^{(j)}(t))_{1 \leq i, j \leq 2}
\]

**Question:** Can a pseudo-variogram have the form

\[
\gamma(t + h, t) = \begin{pmatrix}
\|h\|^{\alpha} & ? \\
? & \|h\|^{\beta}
\end{pmatrix}, \quad 0 < \alpha \neq \beta \leq 2?
\]
What does a pseudo-variogram look like?

\[ \gamma(s, t) = (\text{Var}(W^{(i)}(s) - W^{(j)}(t)))_{1 \leq i, j \leq 2} \]

**Question:** Can a pseudo-variogram have the form

\[ \gamma(t + h, t) = \begin{pmatrix} \|h\|^{\alpha} & \? \\ \? & \|h\|^{\beta} \end{pmatrix}, \quad 0 < \alpha \neq \beta \leq 2? \]

**Answer:** No!
What does a pseudo-variogram look like? (cont’d)

**Theorem (Oesting et al., 2013)**

Let $\gamma(s, t)$ a pseudo-variogram that depends on $s - t$ only. Then, $\gamma$ is of the form

$$
\sqrt{\gamma(t + h, t)} = \begin{pmatrix}
\sqrt{\gamma^*(h)} & \sqrt{\gamma^*(h)} \\
\sqrt{\gamma^*(h)} & \sqrt{\gamma^*(h)}
\end{pmatrix} + \begin{pmatrix}
f_{11}(h) & f_{12}(h) \\
f_{21}(h) & f_{22}(h)
\end{pmatrix}, \quad t, h \in \mathbb{R}^d,
$$

for some univariate variogram $\gamma^*$ and bounded functions $(f_{ij}(\cdot))_{1 \leq i, j \leq 2}$. 

![Graphs showing $\gamma_{11}$ and $\gamma_{22}$](image-url)
Construction Principle:

- $Y(\cdot)$: univariate Gaussian process with stationary increments and variogram $\gamma^*$
- $V(\cdot) = (V^{(1)}(\cdot), V^{(2)}(\cdot))$: bivariate stationary Gaussian process with covariance function
  
  $$C(h) = \begin{pmatrix} C_{11}(h) & C_{12}(h) \\ C_{21}(h) & C_{22}(h) \end{pmatrix}$$

$W(\cdot) = (Y(\cdot) + V^{(1)}(\cdot), Y(\cdot) + V^{(2)}(\cdot))$ has a pseudo-variogram

$$\gamma(h) = (\gamma^*(h) + C_{ij}(0) + C_{jj}(0) - 2C_{ij}(h))_{i,j=1,2}.$$
Reminder:

- Marginals of $V_{\text{max}}^\text{obs}$ and $V_{\text{max}}^\text{pred}$ are modelled by GEVs (parameters estimated MLE).

\[ V_{\text{obs max}} \text{(GEV)} \quad V_{\text{pred max}} \text{(GEV)} \]
Reminder:
- Marginals of $V_{\text{max}}^{\text{obs}}$ and $V_{\text{max}}^{\text{pred}}$ are modelled by GEVs (parameters estimated MLE).
- Data are transformed to standard Gumbel margins ($\sim X^{\text{obs}}, X^{\text{pred}}$).
Reminder:
marginals of $V_{\text{max}}^{\text{obs}}$ and $V_{\text{max}}^{\text{pred}}$ are modelled by GEVs (parameters estimated MLE)

data are transformed to standard Gumbel margins ($\sim X^{\text{obs}}, X^{\text{pred}}$)

standardized observation and forecast are jointly modelled by bivariate BR process
  - dependence in space
  - dependence between observations and forecast
Bivariate Variogram Model

$$\gamma(h) = (\gamma^*(h) + C_{ii}(0) + C_{jj}(0) - 2C_{ij}(h))_{i,j=1,2}.$$ 

- $$\gamma^*$$: variogramm of power law type

$$\gamma^*(h) = \frac{||h||^2}{(1 + ||h||^2)\beta}, \quad \beta \in (0, 1)$$

![Graph showing variograms for different values of \(\beta\).](image)
Bivariate Variogram Model

\[ \gamma(h) = (\gamma^*(h) + C_{ii}(0) + C_{jj}(0) - 2C_{ij}(h))_{i,j=1,2}. \]

- \( C \): bivariate Matérn model (Gneiting et. al., 2010)

\[
C_{ij}(h) = \rho_{ij} \sigma_i \sigma_j \frac{2^{1-\nu_{ij}}}{\Gamma(\nu_{ij})} (a \| h \|)^{\nu_{ij}} K_{\nu_{ij}}(a \| h \|)
\]

for \( \sigma_1, \sigma_2 \geq 0, a, \nu_{11}, \nu_{22} > 0, \nu_{12} = (\nu_{11} + \nu_{22})/2 \) and suitable \( \rho_{ij} \)
Bivariate Variogram Model

\[ \gamma(h) = (\gamma^*(h) + C_{ii}(0) + C_{jj}(0) - 2C_{ij}(h))_{i,j=1,2}. \]

- \( \gamma^* \): \[ \gamma^*(h) = \frac{\|Ah\|^2}{(1 + \|Ah\|^2)^\beta}, \quad \beta \in (0, 1) \]
- \( C \): \[ C_{ij}(h) = \rho_{ij}\sigma_i\sigma_j2^{1-\nu_{ij}}\Gamma(\nu_{ij})^{-1}(a\|Ah\|)^{\nu_{ij}}K_{\nu_{ij}}(a\|Ah\|) \]
- introduce anisotropy matrix \( A \) (dilation/rotation)
Estimation of Dependence Structure

Extremal coefficient function for bivariate processes:

\[
\begin{align*}
\mathbb{P}(X^{\text{obs}}(s) \leq x, X^{\text{obs}}(t) \leq x) &= \mathbb{P}(X^{\text{obs}}(0) \leq x)^{\theta^{\text{obs,obs}}(s,t)} \\
\mathbb{P}(X^{\text{obs}}(s) \leq x, X^{\text{pred}}(t) \leq x) &= \mathbb{P}(X^{\text{pred}}(0) \leq x)^{\theta^{\text{obs,pred}}(s,t)} \\
\mathbb{P}(X^{\text{pred}}(s) \leq x, X^{\text{obs}}(t) \leq x) &= \mathbb{P}(X^{\text{obs}}(0) \leq x)^{\theta^{\text{pred,obs}}(s,t)} \\
\mathbb{P}(X^{\text{pred}}(s) \leq x, X^{\text{pred}}(t) \leq x) &= \mathbb{P}(X^{\text{pred}}(0) \leq x)^{\theta^{\text{pred,pred}}(s,t)}
\end{align*}
\]

Estimation: components are estimated separately via $F$-madogram

ECF for bivariate BR processes:

\[
\begin{pmatrix}
\theta^{\text{obs,obs}}(s,t) & \theta^{\text{obs,pred}}(s,t) \\
\theta^{\text{pred,obs}}(s,t) & \theta^{\text{pred,pred}}(s,t)
\end{pmatrix}
\begin{pmatrix}
\sqrt{\gamma_{ij}(s-t)}
\end{pmatrix}
= 
2\Phi \left( \frac{\sqrt{\gamma_{ij}(s-t)}}{2} \right)
\]

\[i,j=1,2\]
Extremal Coefficient Function

ECF between $X_{\text{obs}}^{\text{obs}}$ and $X_{\text{obs}}^{\text{obs}}$

ECF between $X_{\text{obs}}^{\text{obs}}$ and $X_{\text{pred}}^{\text{pred}}$

ECF between $X_{\text{pred}}^{\text{pred}}$ and $X_{\text{pred}}^{\text{pred}}$
Unconditional simulation of the Brown-Resnick process:

Realisation of $X^{\text{obs}}$

Realisation of $X^{\text{pred}}$
Outlook: Post-Processing of the Forecast

Given Data

- historical observation and forecast data
- forecast for today
Outlook: Post-Processing of the Forecast

Given Data

- historical observation and forecast data
- forecast for today

Bivariate BR process provides a model for observed wind gusts conditional on forecast

\[ \sim \text{post-processed forecast} \]
Outlook: Post-Processing of the Forecast

**Given Data**
- historical observation and forecast data \(\rightarrow\) estimation of \(\xi, \mu, \sigma, \gamma\)
- forecast for today \(\rightarrow\) estimation of “weather parameters”

Bivariate BR process provides a model for observed wind gusts conditional on forecast
\(\rightarrow\) post-processed forecast
Outlook: Post-Processing of the Forecast

Given Data

- historical observation and forecast data $\leadsto$ estimation of $\xi, \mu, \sigma, \gamma$
- forecast for today $\leadsto$ estimation of “weather parameters”

Bivariate BR process provides a model for observed wind gusts conditional on forecast $\leadsto$ post-processed forecast

1. standardized forecast $V^{\text{pred}}_{\max}$ to standard Gumbel margins $\leadsto X^{\text{pred}}$
2. simulate realizations of $X^{\text{pred}} \mid X^{\text{obs}}$
3. transform $X^{\text{obs}}$ from Gumbel to GEV margins $\leadsto V^{\text{obs}}_{\max}$