Modelling multivariate counts varying continuously in space

Alexandra M. Schmidt
IM - UFRJ, Brazil

Homepage: www.dme.ufrj.br/alex

joint work with Marco A. Rodríguez
UQTR, Canada

in Bayesian Statistics 9, pp. 611-628

PASI 2014, Workshop on Multivariate Spatial Stats

Búzios, Brazil, June 2014
Outline of the talk

- Motivation and aims
- Distributions for multivariate counts: a brief review
- Our proposed model
- Covariates in covariance structure
- Model comparison and results
- Discussion
On each sampling date, measurements were made at a cluster of locations on a shore, selected randomly among all clusters on that shore.

Sampling dates were unevenly spaced in time over a period of 70 days; the North and South shores were visited in alternation on consecutive sampling dates.
Fish were collected by electrofishing: fish are attracted to anodes hanging from the booms.
Most abundant fish species observed in Lake St. Pierre

(a) Yellow perch

(b) Brown bullhead

(c) Golden shiner

(d) Pumpkinseed
Main aims

- Assess the influence of local habitat (characterized by environmental covariates water depth, water transparency, substrate, vegetation) on the abundance of fish species
- Determine whether species abundances are correlated across space and among themselves
- Understand the spatial distribution of each species
Distributions for multivariate counts

Probability distributions for multivariate counts can be defined through:

the sum of independent Poisson distributions \(\rightarrow\) does not account for overdispersion or negative covariance;
Distributions for multivariate counts

Probability distributions for multivariate counts can be defined through:

continuous mixtures of independent Poisson distributions. Let \( Y = (Y_1, \cdots, Y_K) \), with \( Y_k | \delta_k \sim \text{Poi}(\delta_k) \), \( k = 1, \cdots, K \), conditionally independent, then

\[
f(y | \theta) = \int \prod_{k=1}^{K} p(y_k | \delta_k) g(\delta | \theta) d\delta,
\]

where \( \delta = (\delta_1, \cdots, \delta_K) \).

- \( g(.) \) can be the pdf of a multivariate gamma \( \rightarrow \) accounts for overdispersion but not for negative covariance
- \( g(.) \) can be the pdf of a multivariate log-normal distribution
Multivariate Poisson as a Sum of Independent Poisson Random Variables

Let

\[
Y_1 = W_1 + W_{12} + W_{13} + W_{123}
\]
\[
Y_2 = W_2 + W_{12} + W_{23} + W_{123}
\]
\[
Y_3 = W_3 + W_{23} + W_{13} + W_{123},
\]

with \(W_i \sim \text{Poi}(\lambda_i), W_{ij} \sim \text{Poi}(\lambda_{ij}) W_{ijl} \sim \text{Poi}(\lambda_{ijl}),\)
\(i, j, l = \{1, 2, 3\}, i < j < l.\)

More generally, \(Y = BW,\) such that, \(E(Y) = B\lambda\) and \(V(Y) = B\Sigma B^T,\) where \(\Sigma = \text{diag} (\lambda_1, \cdots, \lambda_q).\)

- Mean and variance of \(Y_i\) are assumed identical.
- Model only accounts for positive covariance structures.
Multivariate Count Distributions Based on Mixtures

Let \( Y = (Y_1, \cdots, Y_K) \), with \( Y_k \mid \delta_k \sim \text{Poi}(\delta_k) \), \( k = 1, \cdots, K \), conditionally independent, then

\[
f(y \mid \theta) = \int \prod_{k=1}^{K} p(y_k \mid \delta_k) g(\delta \mid \theta) d\delta,
\]

where \( \delta = (\delta_1, \cdots, \delta_K) \).
Multivariate Poisson-Gamma Mixture

A multivariate gamma distribution for a $K$-dimensional random vector $\delta$ can be obtained by defining

$$\delta = B W = \begin{pmatrix} \frac{b_0}{b_1} & 1 & 0 & \cdots & 0 \\ \frac{b_0}{b_2} & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{b_0}{b_K} & 0 & 0 & \cdots & 1 \end{pmatrix} \begin{pmatrix} W_0 \\ W_1 \\ \vdots \\ W_K \end{pmatrix}, \quad (1)$$

with $W_k \sim Ga(a_k, b_k)$ for $k = 0, 1, 2, \cdots, K$; $W_k$ independent among themselves (Mathai & Moschopoulos, 1991).

It is easy to show that $\text{Cov}(\delta_k, \delta_l) = \frac{a_0}{b_kb_l}$, $k, l = 1, 2, \cdots, K$. 
If $\mathbf{Y} = (Y_1, \cdots, Y_K)$, such that $Y_i|\delta_i \sim \text{Poi}(\delta_i)$, independently, and $\delta = \mathbf{BW}$ it can be shown that, marginally,

$$E(Y_i) = \frac{a_i}{b_i} + \frac{a_0}{b_i} = \mu_i \quad i, j = 1, \cdots, K$$

$$V(Y_i) = \mu_i + \frac{a_0}{b_i^2} + \frac{a_i}{b_i}$$

$$\text{Cov}(Y_i, Y_j) = \frac{a_0}{b_i b_j}.$$ 

- It accounts for overdispersion,
- however, it only captures positive covariance structures.
Multivariate Poisson-lognormal mixture (Aitchison & Ho 1989)

Assume that

$$\log \delta \sim N_K(\mu, \Sigma),$$

where $\Sigma$ has elements $\sigma_{kl}$. Assuming $Y_k \mid \delta_k \sim \text{Poi}(\delta_k)$ independently.

Marginally,

$$E(Y_k) = \exp \left( \mu_k + \frac{1}{2} \sigma_{kk} \right) = \alpha_k$$

$$V(Y_k) = \alpha_k + \alpha_k^2 \{ \exp(\sigma_{kk}) - 1 \}$$

$$\text{Cov}(Y_k, Y_l) = \alpha_k \alpha_l \{ \exp(\sigma_{kl}) - 1 \}, \ k, l = 1, \ldots, K.$$

- This model accounts for overdispersion
- Allows for both positive and negative covariance structures
Multivariate Poisson-lognormal mixture (Aitchison & Ho 1989)

Assume that

$$\log \delta \sim N_K(\mu, \Sigma),$$

where $\Sigma$ has elements $\sigma_{kl}$. Assuming $Y_k | \delta_k \sim \text{Poi}(\delta_k)$ independently.

This approach is appealing, as we can write

$$\log \delta_k = \beta X_k + \epsilon_k, \ k = 1, \ldots, K$$

$$\epsilon \sim N_K(0, \Sigma)$$

- Chib and Winkelmann (2001) were the first to provide a full Bayesian treatment of this model
- A similar approach has been widely used for multiple disease mapping (e.g., Carlin and Banerjee 2003, Gelfand and Vounatsou 2003, Jin et al. 2007; good overview in Lawson 2009)
Proposed model

Let $Y_k(s_{tj})$ represent the number of individuals (counts) of species $k$, $k = 1, \cdots, K$, observed at location $s_{tj}$, $j = 1, \cdots, n_t$ and time $t = 1, \cdots, T$. We assume

$$Y_k(s_{tj}) \mid \theta_k(s_{tj}), \delta_k(s_{tj}) \sim \text{Poi}(\theta_k(s_{tj})\delta_k(s_{tj})),$$

where

$$\log \theta_k(s_{tj}) = X_k(s_{tj})\beta_k$$

which captures local habitat structures.

The parameter vector $\delta(s_{tj}) = (\delta_1(s_{tj}), \cdots, \delta_K(s_{tj}))'$ plays the role of the mixing component.
Proposed model - Mixing component

\[ \log \delta_k(s_{tj}) = \gamma_k(s_t) + \nu_k(s_{tj}), \]

\[ \log(\text{mixing component}) = \text{temporal effect} + \text{local effect} \]
Multivariate counts in space
Alex Schmidt
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Multivariate Spatial Stats
Búzios, Brazil, 2014

Outline
Motivation and aims
Distr. multiv. counts
Proposed model
Inference procedure
Results
Model comparison
Results
Conclusions
Main references

Proposed model - Mixing component

\[ \log \delta_k(s_{tj}) = \gamma_k(s_t) + \nu_k(s_{tj}), \]

Modelling the temporal effect:

Let \( \gamma(s_t) = (\gamma_1(s_t) \cdots \gamma_K(s_t))' \).

We assume

\[ \gamma(s_t) \sim N_K(0, \Omega), \quad \forall t = 1, \cdots, T, \]

where \( \Omega \) captures the covariance among species at each time \( t \).
Proposed model - Mixing component

\[
\log \delta_k(s_{t_j}) = \gamma_k(s_t) + \nu_k(s_{t_j}),
\]

Modelling the local effect:
Let \( \nu(s_{t_j}) = (\nu_1(s_{t_j}), \cdots, \nu_K(s_{t_j})) \), following the LMC (Gelfand et al 2004)

\[
\nu(s_{t_j}) = A\omega(s_{t_j}).
\]

\( A \) is lower triangular and \( \omega(s_{t_j}) = (\omega_1(s_{t_j}) \cdots \omega_K(s_{t_j}))' \)
such that, independently,

\[
\omega_k(.) \sim GP(0, \rho(\theta_k; d)), \quad k = 1, \cdots, K.
\]

Then

\[
\text{Cov}(\nu(s), \nu(s')) = \sum_{k=1}^{K} \rho(\theta_k; d)M_k, \quad \text{with } M_k = a_k a_k^T
\]
Joint distribution of $\log \delta$

Let $\delta$ be the $nK$-dimensional vector containing the elements of the mixing component, 
\[ \delta = (\delta(s_{1}) , \cdots , \delta(s_{nT}))'. \]

- **Separable model** ($\rho(\vartheta; d)$, for $k = 1, \cdots , K$)

\[ \log \delta | \gamma \sim N((I_{K} \otimes C)\gamma , R \otimes M), \]

where $M = AA^T$.

- **Non-separable model** ($\rho(\vartheta_{k}; d)$)

\[ \log \delta | \gamma \sim N((I_{K} \otimes C)\gamma , \sum_{j=1}^{K} R_{j} \otimes M_{k}). \]

We now discuss the specification of the spatial correlation function (elements of $R$).
Joint distribution of $\log \delta$

Let $\delta$ be the $nK$-dimensional vector containing the elements of the mixing component, $\delta = (\delta(s_{11}), \cdots, \delta(s_{Tn_T}))'$.

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We now discuss the specification of the spatial correlation function (elements of $R$).
Isotropic model

The correlation structure of $\omega_k(.)$ is given by

$$\rho(s, s'; \vartheta_k) = \exp(-\phi_k ||s - s'||).$$

with parameter $\vartheta_k = \phi_k$. 
The correlation structure of $\omega_k(.)$ is given by

$$\rho(s, s'; \vartheta_k) = \exp(-\phi_k \| d_k(s) - d_k(s') \|),$$

with $d_k(s) = s D_k$, and

$$D_k = \begin{bmatrix} \cos \psi_{A_k} & -\sin \psi_{A_k} \\ \sin \psi_{A_k} & \cos \psi_{A_k} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & \psi_{R_k}^{-1} \end{bmatrix},$$

where $\psi_{A_k}$ is the anisotropy angle and $\psi_{R_k} > 1$ is the anisotropy ratio, with parameters $\vartheta_k = (\phi_k, \psi_{A_k}, \psi_{R_k}).$
Covariates in the covariance structure (Schmidt et al. 2011)

Now location has a broader interpretation. Let \( w = (\text{long}, \text{lat}, z_3, \cdots, z_C) \), and

\[
\rho(w, w'; \theta_k) = \exp \left( -\sqrt{(w - w')^T \Phi^{-1} (w - w')} \right).
\]

Similar to assuming a GP in \( \mathbb{R}^C \), generating an anisotropic correlation function in \( \mathbb{R}^2 \) (“geographical” space).

Our model assumes

\[
\rho(w, w'; \theta_k) = \exp \left( -\phi_1 ||s - s'|| - \phi_2 |z - z'| \right),
\]

where \( z \) is geodetic depth (because lake level fluctuates, local habitat travels, but geodetic depth is fixed).
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where \( z \) is \textit{geodetic depth} (because lake level fluctuates, local habitat \textit{travels}, but geodetic depth is fixed).
Comparing correlation structures

Isotropic

Elliptical

Covariate

Covariate
Likelihood function

Let

\[ y = (y(s_{11}), y(s_{12}), \ldots, y(s_{1n_1}), \ldots, y(s_{T1}), \ldots, y(s_{Tn_T}))' \]

be the observed counts over the sampling period at each location \( s_{tj}, t = 1, \ldots, T, j = 1, \ldots, n_t \).

The likelihood function is

\[
l(y \mid \theta, \delta) \propto \prod_{t=1}^{T} \prod_{j=1}^{n_t} \prod_{k=1}^{K} \exp \left\{ -\theta_k(s_{tj}) \delta_k(s_{tj}) \right\} \left[ \theta_k(s_{tj}) \delta_k(s_{tj}) \right]^{y_k(s_{tj})}.
\]
Prior specification

- $\beta \sim N(0, \sigma^2_\beta I_p)$
- $\phi_{ij} \sim IG(2, b)$, $i = 1, 2, j = 1, \ldots, K$, and $b$ is fixed such that the practical range (correlation $= 0.05$) is reached at half of the maximum distance between observations.
- $\Omega \sim IW(K + 1, c I_k)$
- $\psi_R \sim \text{Pareto}(1, 2)$, $\psi_A \sim U(0, \pi)$
- $a_{ij} \sim N(0, 5)$ and $\log a_{ii} \sim N(0, 5)$
MCMC sampling scheme

We reparametrize the model such that
\[ \varphi_k(s_j) = \theta_k(s_j) \delta(s_j), \text{ and} \]
\[ \log \varphi_k(s_j) = X^*_k(s_j) \beta^*_k + W_k(s_j) \]
and \( W_k(s_j) = \beta_1 + \gamma_k(s_t) + \nu_k(s_j) \), where
\( X^*_k(.) \) does not have a column of ones, and
\( \beta^*_k = (\beta_{2k}, \ldots, \beta_{pk})^T \).

- \( \beta_k \) and \( W_k(.) \) M-H algorithm proposed by Gamerman (1997)
- \( \beta_1 = (\beta_{11}, \ldots, \beta_{1K})' \) and \( \gamma \) normal posterior full conditionals
- \( \Omega \) follows an inverse-Wishart posterior full conditional
- \( \phi_1, \phi_2, \psi_R, \psi_A \), and elements of \( A \) are sampled through M-H steps
**Fitted models**

**M1:** Separable isotropic covariance structure

**M2:** Separable elliptical anisotropy covariance structure

**M3:** Separable covariate-dependent \((z = \text{geodetic depth})\) covariance structure

**M4:** Non-separable isotropic covariance structure

**M5:** Non-separable elliptical anisotropy covariance structure

**M6:** Non-separable, covariate-dependent \((z = \text{geodetic depth})\) covariance structure

- \(\gamma(s_t)\) assumed to be independent across time and common to both shores
- We allow for different local random effects for the North and South shores (Glémet and Rodríguez 2007; Schmidt et al. 2010b)
M1: Separable isotropic covariance structure

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### Fitted models

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<th>Model</th>
<th>Description</th>
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### Model comparison

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<thead>
<tr>
<th>Model</th>
<th>$p_D$</th>
<th>DIC</th>
<th>EPD</th>
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<tbody>
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<td>M6</td>
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### Yellow perch

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<tbody>
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<td>M3</td>
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<td>133755.5</td>
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<td>133768.1</td>
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<td>133770.4</td>
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<td>M6</td>
<td>101.1</td>
<td>622.9</td>
<td>133754.1</td>
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### Brown bullhead

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<td>M6</td>
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<td>590.6</td>
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### Golden shiner

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<td>54044.9</td>
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<td>429.8</td>
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<td>M3</td>
<td>75.6</td>
<td>428.9</td>
<td>54037.7</td>
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<tr>
<td>M4</td>
<td>82.9</td>
<td>434.0</td>
<td>54050.2</td>
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<tr>
<td>M5</td>
<td>82.8</td>
<td>434.7</td>
<td>54046.9</td>
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<tr>
<td>M6</td>
<td>75.7</td>
<td>432.7</td>
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### Pumpkinseed

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<tr>
<th>Model</th>
<th>$p_D$</th>
<th>DIC</th>
<th>EPD</th>
</tr>
</thead>
<tbody>
<tr>
<td>M1</td>
<td>144.8</td>
<td>1039.0</td>
<td>2019063.4</td>
</tr>
<tr>
<td>M2</td>
<td>145.2</td>
<td>1038.6</td>
<td>2019061.1</td>
</tr>
<tr>
<td>M3</td>
<td>137.1</td>
<td>1031.3</td>
<td>2019023.1</td>
</tr>
<tr>
<td>M4</td>
<td>145.1</td>
<td>1040.0</td>
<td>2019030.6</td>
</tr>
<tr>
<td>M5</td>
<td>144.0</td>
<td>1036.4</td>
<td>2019067.2</td>
</tr>
<tr>
<td>M6</td>
<td>136.5</td>
<td>1032.1</td>
<td>2018988.0</td>
</tr>
</tbody>
</table>

Table: Values of the effective number of parameters, $p_D$, DIC (Spiegelhalter et al. 2001), and EPD (Gelfand and Ghosh, 1998) for the six fitted models, by fish species.

Models including geodetic depth as a covariate in the correlation structure of the spatial effects generally fit better than those assuming isotropy or geometrical anisotropy.
Posterior predictive distributions for counts of the four species under M6 showed agreement with observed counts.
Water transparency had positive influence on the abundance of three of the four fish species, whereas water depth and vegetation each influenced the abundance of one fish species. Substrate composition had no apparent effect on fish abundances.
Variances and correlations tended to be greater in the North shore than in the South shore. The two strongest correlations, both in the North shore, were positive. Negative correlations, possibly indicative of competition between species, were not apparent.
For all non-separable models, the decay parameter $\phi_1$ showed well-defined information gain and marked differences among component spatial processes $\omega_k(.)$. 
M5 provided strong evidence of anisotropy associated with spatial process $\omega_1(.)$, but little deviation from the priors for components $\omega_2(.)$, $\omega_3(.)$, and $\omega_4(.)$, indicating a single anisotropic pattern shared across species and suggesting that the spatial component of M5 may be overparametrized.
Similar to M5, the posterior distribution of $\phi_2$ under M6 provides strong evidence of anisotropy. However, in contrast to M5, the gain in information reflected in differences between priors and posteriors is distributed across components, indicating that species do not share a common anisotropic pattern.
Decay of correlation under M6 had directionality generally similar to that of M5, but was further modified by geodetic lake depth.
Multivariate counts in space
Alex Schmidt
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Brazil

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Geodetic depth

Easting

Northing

5114 5118 5122 5126
658 660 662 664 666

Estimated correlation

Euclidean distance (km)

M4
M5
M6

Yellow perch

Brown bullhead

Euclidean distance (km)
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Separable models

Yellow perch

Brown bullhead

Golden shiner

Pumpkinseed

Estimated correlation

Range of 95% C.I.
Conclusions

- Our model accounts for overdispersion and both positive and negative covariances (among species and across space)
- The LMC provided flexible covariance structures for the mixing component
- Anisotropic spatial effects improved fit (DIC and EPD)
- Including information on geodetic lake depth in the spatial covariance structure of the spatial process provided a flexible, yet relatively simple, means of capturing anisotropy along the shorelines of the lake
Main references