Spatio-temporal processes, including dynamic linear models

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1. Introduction

Consider a process $X(s, t)$ varying with $s \in S \subset \mathbb{R}^d$ and $t \in T \subset \mathbb{R}$.

I will be concentrating on $d = 2$ and $T = \{t_1, ..., t_n\}$.

[Theoretically, continuous $T$ simply means that $d \to d + 1$]

2 routes are usually taken for modeling:

- direct (observation-driven)
- latent (parameter-driven)

Routes are somewhat disjoint
2. Direct approach

Covariance function (CF): \( \text{Cov}[X(s, t), X(s', t')] = K(s, s'; t, t') \)

Too general for reliable estimation without massive replications over \( S \times T \)

Simplifying assumptions:

1) Temporal stationarity: \( K(s, s'; t, t') = f(s, s'; t - t') \)

2) Spatial stationarity: \( K(s, s'; t, t') = f(s - s'; t, t') \)

3) Spatio-temporal stationarity: \( K(s, s'; t, t') = f(s - s'; t - t') \)

4) Separability: \( K(s, s'; t, t') = f_1(s, s') f_2(t, t') \)

Cressie and Huang (1999), Gneiting (2002), many others:

stationary, non-separable CF’s
3. Latent approach

Explain $X$ behaviour through model components

CF’s appear as a consequence

Knowledge on how $X$ varies over $(s, t)$, through known functions $\eta_i(s, t)$,

$$\Rightarrow X(s, t) = \sum_i \beta_i \eta_i(s, t) \ ( + e(s, t) )$$

errors $e(s, t)$ do not carry any spatio-temporal correlation

Usually not feasible $\rightarrow$ knowledge rarely exists

Even when it does, there may remain substantial correlation

Idea can be used to span the space of possible representations
\[ X(s, t) = \sum_i \beta_i \eta_i(s, t) \quad (+e(s, t)) \]

Now: unknown \( \eta_i \) functions form a basis for (smooth) functions of \( S \times T \)

These functions adequately chosen to cover the entire space \( S \times T \)

This may require too many \( \eta_i \)’s

Alternative: tensor product

\( \{ \eta_i(s, t) \} \) replaced by \( \{ \phi_i(s) \} \) and \( \{ \alpha_j(t) \} \)

\[ X(s, t) = \sum_i \phi_i(s) \alpha_i(t) \quad (+e(s, t)) \]

Substantial dimension reduction, but there is a cost

More importantly, relevant correlation may still remain
General approach: allow $\{\eta_i(s, t)\}$ to be stochastic

How? to be detailed later

Simplified approach: allow $\{\phi_i(s)\}$ and/or $\{\alpha_j(t)\}$ to be stochastic

$\Rightarrow X(s, t) = \sum_i \phi_i(s) \alpha_i(t) \ ( +e(s, t) )$

$\{\phi_i(s)\}$ are Gaussian processes (GP) and/or $\{\alpha_j(t)\}$ are time series

Usual time series model: autoregressive

Example - AR(1): $\alpha(t) = G(t) \alpha(t - 1) + w(t)$, $w(t)$ iid $N(0, \Sigma)$

Lots of references: Mardia, Goodall, Redfern & Alonso (1998), Wikle & Cressie (1999), Stroud, Muller & Sansó (2001), Calder (2007), Sansó, Schmidt & Nobre (2008), ...

Also leads to non-separable CF’s
4. Dynamic Gaussian processes

Set $\eta(s, t) = (\eta_1(s, t), \ldots, \eta_m(s, t))$

DGP: $\eta(s, t) = G(t) \eta(s, t - 1) + w(s, t), \quad w(\cdot, t) \quad iid \quad mGP(0, \rho)$

Process is completed with initialization $\eta(s, 1) \sim mGP$

Leads to non-separable but now also temporally non-stationary CF's

Simplest DGP: univariate spatio-temporal random walk

$\eta(s, t) = \eta(s, t - 1) + w(s, t), \quad w(\cdot, t) \quad iid \quad GP$

Useful model for smooth spatio-temporal variation of the data $Y(s, t)$

$Y(s, t) = \eta(s, t) + WN \quad error$

Can handle spatio-temporal heterogeneity of other model components
4.1. Regression

Assume the presence of covariates $Z(s, t)$ associated with data $Y(s, t)$

Standard approach: $Y(s, t) = Z(s, t)^T \eta + X_0(s, t) + WN$ error

with $X_0 \sim DGP$

Spatio-temporal heterogeneity may also be present in the regression part

Revised approach: $Y(s, t) = Z(s, t)^T \eta(s, t) + X_0(s, t) + WN$ error

with $X = [\eta, X_0] \sim DGP$

The mean of the DGP may be separated from $X(s, t)$

Now, $X(s, t) = X(t) + X^*(s, t)$ where $X^*$ is a zero-mean DGP

Model completed with an evolution for time-varying mean $X(t)$
4.2. Trend

Assume the DGP $X(s, t)$ is subject to spatio-temporal variations

These are modeled with an auxiliary growth process $\gamma(s, t)$

$$X(s, t) = X(s, t - 1) + \gamma(s, t) + w_X(s, t), \quad w_X(\cdot, t) \text{ iid } GP$$

$$\gamma(s, t) = \gamma(s, t - 1) + w_\gamma(s, t), \quad w_\gamma(\cdot, t) \text{ iid } GP$$

This is in the form of a bivariate DGP with $G(t) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$

Acceleration and higher order variations can be equally defined

Useful for estimation of trend features and specially for prediction
4.3. Seasonality

Assume the DGP $X(s, t)$ is subject to seasonal variations

Seasonality given by a sinusoidal wave (temperature): single harmonic

Requires an auxiliary process $\gamma(s, t)$

$$
\begin{pmatrix}
X(s, t) \\
\gamma(s, t)
\end{pmatrix}
= 
\begin{pmatrix}
\cos(2\pi/q) & \sin(2\pi/q) \\
-\sin(2\pi/q) & \cos(2\pi/q)
\end{pmatrix}
\begin{pmatrix}
X(s, t - 1) \\
\gamma(s, t - 1)
\end{pmatrix}
+ w_S(s, t),
$$

where $w_S(\cdot, t)$ iid $2GP$ and $q$ is the seasonal cycle length.

More elaborate seasonal patterns $\rightarrow$ additional bivariate processes

$(X_2, \gamma_2), \ldots, (X_{[q/2]}, \gamma_{[q/2]})$ associated with 2nd, ..., $[q/2]$th harmonics

Simpler form: $\sum_{j=1}^{q} X(s, t - j) = w(s, t)$, where $w(\cdot, t)$ iid $1GP$

but requires $q$-dimensional DGP
5. Applications

DGP can be used in models for 3 types of spatial data

Directly applied to the mean structure of spatio-temporal data $Y(s, t)$

5.1. Geostatistics - continuous space

Spatial version of generalized dynamic linear models

\[ Y(s, t) \sim \mathcal{F} \text{ with mean } \mu(s, t) \quad \left[ \text{eg } N\left( \mu(s, t), \sigma^2 \right) \right] \]

\[ g[\mu(s, t)] = F(s, t)^T X(s, t) \quad \left[ F^T = (Z^T, 1) \text{ and } X = (\eta, X_0) \right] \]

\[ X(s, t) = G(t) X(s, t - 1) + w_X(s, t), \quad w_X(\cdot, t) \ \text{iid} \ \text{GP} \]
Illustration: Effect of precipitation ($\text{Prec}$) over temperature ($\text{Temp}$)

\[
\text{Temp}(s, t) = \beta_0(s, t) + \beta_1(s, t) \text{Prec}(s, t) + v(s, t)
\]

\[
\beta_0(s, t) = \beta_0(s, t - 1) + w_0(s, t), \quad w_0(\cdot, t) \text{ iid } GP
\]

\[
\beta_1(s, t) = \beta_1(s, t - 1) + w_1(s, t), \quad w_1(\cdot, t) \text{ iid } GP
\]

Intercept $\beta_0$ and regression coef $\beta_1$ are spatio-temporally varying

Disturbance processes $w_0$ and $w_1$ may be related

eg. linear transformations of independent GP’s
\( S: \) region in the state of Colorado, USA ( + - monitoring stations )

\( T: \) \( \{ \text{Jan/1997, \cdots, Dec/1997} \} \)
Posterior mean of the spatio-temporal variation of the regression coefficient $\beta_1$ for a region of the State of Colorado, USA (Gelfand, Banerjee and Gamerman, 2005).
5.2. Areal data

[based on Vivar and Ferreira (JCGS, 2012)]

Correlations are no longer based on CF’s and Euclidean space

Mostly based on precision (inv. variance) matrices and neighbourhoods

Dynamic GP’s replaced by dynamic Markov random fields (MRF)

Generalized spatio-temporal linear model

\[ Y_{i,t} \sim F \text{ with mean } \mu_{i,t} \quad \left[ \text{eg } N \left( \mu_{i,t}, \sigma^2 \right) \right] \]

\[
 g[\mu_{i,t}] = Z(s,t)^T X_{i,t}
\]

\[
 X_t = G(t)X_{t-1} + w_t, \quad w_t \text{ iid MRF}(0,Q) \equiv N(0,Q^{-1})
\]

\[ Q \text{ is sparse (full of 0’s) } \quad q_{ij} \neq 0 \text{ indicates neighborhood } (i,j) \]
Illustration:

Evolution of the homicide rate in Rio de Janeiro state municipalities

Contamination model

\[ Y_{i,t} = \mu_{i,t} + e_{i,t}, \text{ where } e_{i,t} \sim N(0, \sigma^2_{i,t}) \]

\[ \mu_t = H\mu_{t-1} + w_t, \text{ where } w_t \sim N(0, Q^{-1}) \]

\[ h_{i,j} = c \times \begin{cases} 1, & \text{if } i = j \\ \alpha, & \text{if } i \text{ and } j \text{ are neighbors} \\ 0, & \text{otherwise} \end{cases} \]

\( \alpha \) is the contamination index

\( c \) is the contamination persistence
Posterior mean of the area level
5.3. Point pattern

based on Pinto Jr. (2014)

Data: space-time locations of occurrence of events

Usual models: PP, NHPP, Cox process (CP), log-Gaussian CP

LGCP (space only): \( Y \sim \text{NHPP}(\lambda) \) and \( \log \lambda = X \sim \text{GP} \)

Liang et al (2008): \( \log \lambda(s) = Z(s)^T \eta + X(s) \), \( X \sim \text{GP} \)

Our extension:

\[
Y \sim \text{PP}(\lambda) \text{ where } \lambda : S \times T \to R^+
\]

\[
g[\lambda(s, t)] = Z(s, t)^T \eta(s, t) \quad [\text{eg. : } g = \log ]
\]

\[
\eta(s, t) = G(t) \eta(s, t - 1) + w_\eta(s, t), \quad w_\eta(\cdot, t) \text{ iid } \text{GP}
\]
Likelihood:

\[ l(\lambda; Y) = \prod_i \lambda(s_i, t_i) \prod_t \exp \left[ - \int_S \lambda(u, t) \, du \right] \]

difficult problem \(\rightarrow\) depends on an infinite-dimensional unknown function

Møller et al (1998) solution for CP: discretize \(\lambda\) over space \(\rightarrow\)

problem: it is an approximation

Adams et al (2009) solution for CP: use thinning \(\rightarrow\)

problems: does not extend easily for discrete time and for further model components and dimension explosion and MCMC...

We are currently working on these solutions for our DGP model
Illustration:

Evolution of the vehicle casualties (theft/robbery) in Rio de Janeiro city considering Type (private/commercial) and Age (manufacturing year)

\[ Y \sim PP(r, \lambda) \text{ where } r, \lambda : S \times T \rightarrow R^+ \]

\[ r(s, t; \text{Type}, \text{Age}) = \text{total exposure at } (s, t) \quad [\text{known offset}] \]

\[ \lambda(s, t; \text{Type}, \text{Age}) = \exp\{X_0(s, t) + \text{Type } X_1(s, t) + \text{Age } X_2(s, t)\} \]

\[ \{X_0, X_1, X_2\}(s, t) = \{X_0, X_1, X_2\}(s, t - 1) + w_X(s, t), \ w_X(\cdot, t) \text{ iid GP} \]
$S$: Rio de Janeiro municipality

$T$: \{sem1/2009, \cdots, sem2/2011\}
Posterior mean of the spatio-temporal coefficient process of Age
6. Final comments

Flexible approach

Handles non-separable processes

Handles spatial non-stationary/stationary processes

Handles temporal non-stationary/stationary processes

More on the point process: Jony Pinto Jr. (here)

More on point process without discretization: F. B. Gonçalves (JSM 2014)
Thank you!

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