Instructions

- There are six questions on this exam. Answer any five of them. If you attempt more than five, indicate below which five you want graded. If you do not provide such an indication, the first five will be graded.

- Each question is worth 20 points, although the questions vary in difficulty.

- Write your answers in the booklets provided. Use a new page for each question, and do not write on the back of the answer pages. The pages will be duplicated for grading, and if you write on the back, that material will not be graded and may adversely influence your grade. Label the pages at the top of each answer sheet with your ID number, the question number, and the sub-question numbers. If you fill up a booklet, ask for another.

- The examination is closed-book and closed-notes. Calculators and electronic devices are not allowed, and are not needed. The time limit is three hours.

- If you are using a pencil, please use only a number 2 lead pencil, to ensure adequate reproduction results.

Good luck!

Your Exam ID #: ________________________________

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<th>Problem:</th>
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Problem 1. Let $X_1, \ldots, X_n$ be independent identically distributed random variables having density function 

$$f(x) = (\theta + 1)x^\theta 1_{\{0 < x < 1\}}$$

for some unknown $\theta > -1$. In answering the following questions, you may assume that standard regularity conditions hold.

(a) Find a sufficient statistic for $\theta$. Credit will be given according to the extent of data reduction in the sufficient statistic.

(b) Find the Cramér-Rao lower bound for the variance of an unbiased estimator of $\theta$.

(c) For what functions of $\theta$ does an unbiased estimator exist which meets the Cramér-Rao lower bound?

(d) Define $\tau \equiv g(\theta) \equiv E_\theta[X]$, where $X$ is a random variable with density $f$. View the sample mean $\bar{X} \equiv \bar{X}_n$ as an unbiased estimator of $\tau = g(\theta)$ and compute its variance.

(e) Compute the Cramér-Rao lower bound for the variance of an unbiased estimator of $\tau = g(\theta)$.

(f) Show that the efficiency of $\bar{X}$ as an estimator of $\tau = g(\theta)$ is strictly less than 1.
Problem 2. Let $X_1, \ldots, X_m$ be independent identically distributed $\mathcal{N}(\mu, \sigma^2)$, and let $Y_1, \ldots, Y_n$ be independent identically distributed $\mathcal{N}(\nu, a^2 \sigma^2)$, where $a^2$ is a known constant, and where the random vectors $(X_1, \ldots, X_m)$ and $(Y_1, \ldots, Y_n)$ are independent. Note the possibly distinct sample sizes $m$ and $n$.

(a) Define the parameter vector $\theta$ for this problem. Find a three-dimensional sufficient statistic $T$ for this family of distributions. Is the family of distributions of $T$ complete? Answer with yes or no, and give reasons.

(b) Find the joint distribution of the sufficient statistic $T$, or the joint distribution of a convenient one-to-one function of $T$. Use the quickest method you know of. Formal proof is not required here, but quickly summarize a justification.

(c) Find the maximum likelihood estimators for $\mu$, $\nu$, $\sigma^2$, and $(\mu - \nu)/\sigma^2$. Use the quickest method you know. Long formal proofs are not required here, but you are expected to appeal to the classical results for an independent identically distributed sample from some normal distribution.

(d) Find the uniformly minimum variance unbiased estimators (UMVUEs) of $\mu$, $\nu$, and $\sigma^2$. Use the quickest method you know. Again, justify your answers.
Problem 3. Let $Z$ and $V$ denote independent random variables, where $Z$ is $\mathcal{N}(0, 1)$ and $V$ is $\chi^2(m)$.

(a) Write down the joint density of $Z$ and $V$.

(b) Obtain the joint density of $T = Z/\sqrt{V/m}$ and $V$ via the transformation method.

(c) Determine the conditional density of $V$ given that $T = t$ and the marginal density of $T$. (Or, use any method you wish to derive these densities from your expression for the joint density. Both densities are members of famous families of distributions.)
Problem 4. Suppose that $X_1, \ldots, X_n$ are independent identically distributed random variables with probability density function

$$f(x \mid \theta) = e^{\theta x - A(\theta)} h(x),$$

where $\theta$ is an unknown real parameter. The conjugate prior family is given by

$$\pi(\theta \mid n_0, \mu_0) = c_0 e^{n_0(\mu_0 \theta - A(\theta))},$$

where $n_0$ and $\mu_0$ are known and $c_0$ is a normalizing constant. In answering the following questions, you may assume that standard regularity conditions hold.

(a) Show that $E[X] = A'(\theta)$ where $X$ is a random variable with density $f(x \mid \theta)$.

(b) Show that $E[A'(\theta)] = \mu_0$ where the expectation is taken with respect to the prior distribution.

(c) Find the posterior distribution $\pi(\theta \mid x_1, \ldots, x_n)$, and show that it is in the same family as the prior distribution? (Hint: Think of $n_0$ as the prior sample size, and of $\mu_0$ as the prior mean of $A'(\theta)$.)

(d) Assume now that $X_1, \ldots, X_n$ are Gamma($\alpha_0, \beta$), where $\alpha_0$ is known while $\beta$ has a conjugate prior of the above form. (You may choose to specify the parameter $\beta$ so that either $E[X \mid \alpha_0, \beta] = \alpha_0 \beta$ or $E[X \mid \alpha_0, \beta] = \alpha_0 / \beta$. ) Using the results from parts (b) and (c), find the posterior mean of $\beta$. 

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Problem 5. The cells in a $2 \times 2$ table have the probabilities listed at the left below. The basic model for our experiment is to classify an experimental outcome as belonging to one of the four possible cells. This experiment was repeated independently $n$ times. As indicated in the table at the right below, the number of times that each of the four outcomes occurred is given by $\vec{N} \equiv (N_{11}, N_{12}, N_{21}, N_{22})$. The four marginal totals use •-subscripts.

<table>
<thead>
<tr>
<th>$\alpha \beta$</th>
<th>$\alpha(1-\beta)$</th>
<th>$(1-\alpha)\beta$</th>
<th>$(1-\alpha)(1-\beta)$</th>
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<td>$N_{11}$</td>
<td>$N_{12}$</td>
<td>$N_{1•}$</td>
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<td>$N_{21}$</td>
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<td>$N_{2•}$</td>
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(a) Determine the maximum likelihood estimators (MLEs) $\hat{\alpha} \equiv \hat{\alpha}_n$ and $\hat{\beta} \equiv \hat{\beta}_n$ of $\alpha$ and $\beta$, respectively.

(b) Determine the asymptotic joint distribution of $\hat{\alpha}_n$ and $\hat{\beta}_n$, when these estimators are appropriately normalized. (What, if any, assumptions have you made?)

(c) Consider the special case when $\alpha = \beta$ on the left side above. Denote the common value by $\theta$, and determine the MLE, $\hat{\gamma}_n$, of $\gamma \equiv \sqrt{\theta (1-\theta)}$. Also determine the asymptotic distribution of the appropriately normalized random variable $\hat{\gamma}_n$. 
Problem 6. Consider an inference problem for a one-dimensional parameter, $\theta \in \mathbb{R}$. As a result of a Bayesian analysis, we obtain a proper posterior distribution for $\theta$, which is conditional on the data. Let $\mu \in \mathbb{R}$, $\mu_2 \geq 0$ and $\sigma^2 = \mu_2 - \mu^2$ denote the mean, the second moment and the variance of the posterior distribution for $\theta$.

Our task now is to report a decision, $d$, based on the posterior distribution.

(a) A simple loss function for a two-dimensional decision $d = (d_1, d_2)$ is

$$L(\theta, d) = d_2 + \frac{(\theta - d_1)^2}{d_2},$$

where $d_1 \in \mathbb{R}$ and $d_2 > 0$. Find the Bayes rule, that is, find values of $d_1$ and $d_2$ that minimize the Bayes loss, $E[L(\Theta, d)]$, for a given posterior.

(b) Write down a loss function, $L(\theta, d)$, that leads to the posterior mean as the unique Bayes rule.

(c) Similarly, write down a loss function, $L(\theta, d)$, that leads to the second moment, $\mu_2$, of the posterior as the unique Bayes rule.

(d) What is the Bayes rule if $L(\theta, d) = |d - \theta|$ is the absolute error function? Formal proof is not required here, but quickly summarize a justification. (It suffices if you consider the case in which the posterior distribution for $\theta$ has a ‘nice’ density.)