University of Washington
Biostatistics First-Year Theory Exam
Statistics MS Theory Examination
June 13, 2017
9:30 a.m. – 12:30 p.m.
Gowen 201

Instructions

- There are six questions on this exam. Answer any five of them. If you attempt more than five, indicate below which five you want graded. If you do not provide such an indication, the first five will be graded.

- Each question is worth 20 points, *although the questions vary in difficulty.* All questions have multiple parts, which do not typically have equal point value.

- Write your answers in the booklets provided. *Use a new page for each question, and do not write on the back of the answer pages.* The pages will be duplicated for grading, and if you write on the back, that material will not be graded and may adversely influence your grade. If you fill up a booklet, ask for another. Label the pages at the top of each answer sheet with your ID number, the question number, and the sub-question numbers. *DO NOT WRITE YOUR NAME ON THE SOLUTION SHEETS.*

- The examination is closed-book and closed-notes. Calculators and electronic devices are not allowed, and are not needed. If you have these, please turn them in to the proctor. You may pick them up at the end of the exam. Please note that we will not be responsible for these items.

- The time limit is three hours.

- When you are done with the exam, put all of your exam materials (cover sheet, exam, answer booklets, and scratch paper) in the envelope. Stay seated until the proctor indicates that the exam is over.

- If you are using a pencil, please use only a number 2 lead pencil, to ensure adequate reproduction results.

Your exam ID number and Signature _________________________________

Problems to be graded (mark five):

1 2 3 4 5 6
1. Let $X_1, \ldots, X_n$ be i.i.d. $N(\theta, 1)$. Consider the moment generating function

$$g_t(\theta) = E(\exp(tX_1)).$$

(a) Write out an expression for $g_t(\theta)$ in terms of $\theta$ and $t$.

(b) For each $t$, find the UMVU estimate of $g_t(\theta)$.

(c) Determine whether the estimate, as a function of $t$, is actually a moment generating function of some random variable, and if so, what is its distribution?
2. Let $X_1, \ldots, X_n$ be i.i.d. samples from the distribution with density
$f(x; \theta) = \frac{1}{\theta} e^{-x/\theta}$ for $x > 0$, where $\theta > 0$. This density has mean $\theta$ and variance $\theta^2$.

(a) Use the Neyman-Pearson Lemma to find an exact level $\alpha$ test of $H_0 : \theta = 2$ vs.
$H_1 : \theta = 1$. Express your answer in terms of a chi-square ($\chi^2$) distribution.

Note: If $\text{Gamma}(n, \theta)$ denotes the gamma distribution with shape parameter $n$
and scale parameter $\theta$, then $\text{Gamma}(n, \theta) \overset{d}{=} \frac{\theta}{2} \chi^2_{2n}$.

(b) Is your test given in (a) uniformly most powerful (UMP) for the enlarged alternative $\theta < 2$? Why or why not?

(c) Express the power function $\pi(\theta)$ of your test in (a) in terms of a chi-square dis-
tribution. Use this to show that your test is unbiased for the enlarged alternative $\theta < 2$.

(d) Derive the generalized likelihood ratio test (GLRT) (= Wilks’ test) for testing
$H_0 : \theta = 2$ vs $H_2 : \theta \neq 2$. State an approximation to the distribution of the
GLRT statistic under $H_0$ when $n$ is large (no proof necessary).
3. Let $\xi_1, \xi_2, \ldots$ be independent identically distributed Bernoulli random variables with $\Pr[\xi_i = 1] = \Pr[\xi_i = 0] = 1/2$. For $n = 1, 2, \ldots$, define

$$X_n = \sum_{i=1}^{n} \frac{\xi_i}{2^i}.$$ 

Note that $0 \leq X_n < 1$.

(a) What is the distribution of $X_1$? of $X_2$? of $X_n$? (No proof needed.)

(b) Does $\{X_n\}$ converge in distribution? If so, prove it and find the limiting distribution. If not, why not?

(c) Does $\{X_n\}$ converge in probability? Justify your answer.
4. Let \((X, Y)\) be a bivariate random vector with joint pdf \(f(x, y) = \frac{1}{x+y}\) on the range \(0 < x, y < 1, 0 < x + y < 1\).

(a) Show that \(f(x, y)\) is a pdf (integrates to 1) and find the marginal pdf \(f(x)\) of \(X\).
(b) Find the distribution of \(U \equiv X + Y\). Use this to find \(EX\).
(c) Find \(\text{Var}(X)\). Use this to find \(\text{Cov}(X, Y)\).
(d) Find \(E[Y \mid X]\). Is this a decreasing function of \(X\)? Justify your answer.
5. Let \( X_1, \ldots, X_n \) be an i.i.d. sample from an Exponential distribution with rate parameter \( \lambda > 0 \). So, the common density is

\[
f(x; \lambda) = \lambda e^{-\lambda x}, \quad x \geq 0.
\]

Unfortunately, you do not get to see the original sample but only the fractional parts

\[
Y_i = X_i - \lfloor X_i \rfloor \in [0,1), \quad i = 1, \ldots, n.
\]

You wish to estimate \( \lambda > 0 \) based on these fractional parts.

(a) Show that each \( Y_i \) has probability density function

\[
f(y; \lambda) = \frac{\lambda e^{-\lambda y}}{1 - e^{-\lambda}}, \quad 0 \leq y < 1.
\]

Note that \( \{f(y; \lambda) | \lambda > 0\} \) is an exponential family with natural parameter \( \lambda \).

(b) Find \( E_{\lambda}(Y_1) \). What is its range, i.e., what is the set \( \{ E_{\lambda}(Y_1) | 0 < \lambda < \infty \} \)?

(c) Find a minimal sufficient statistic \( T \) for \( (Y_1, \ldots, Y_n) \) (no proof needed).

(d) By referring to results about exponential families or by direct calculation, find the likelihood equation that determines the maximum likelihood estimator (MLE) \( \hat{\lambda}_n = \hat{\lambda}_n(Y_1, \ldots, Y_n) \). (\( \hat{\lambda}_n \) should be a function of \( T \).)

(e) Give an example of a sample \( Y_1, \ldots, Y_n \in (0,1) \) for which the MLE \( \hat{\lambda}_n \) does not exist. Justify your answer.

(f) Show that for any \( \lambda > 0 \), the probability that the MLE \( \hat{\lambda}_n \) exists converges to 1 as \( n \to \infty \).
6. Let \( g(x) \) and \( h(x) \) be known pdfs, each with support \((-1, 1)\) in \( \mathbb{R}^1 \). Denote the mean and variance under \( g \) (respectively, \( h \)) by \( \mu \) and \( \sigma^2 \) (resp. \( \nu \) and \( \tau^2 \)). Let \( X_1, \ldots, X_n \) be an i.i.d. sample from the mixture pdf

\[
f_\lambda(x) = \lambda g(x) + (1 - \lambda) h(x),
\]

where \( 0 < \lambda < 1 \) is unknown.

(a) Find \( \mathbb{E}_\lambda(X_1) \) and \( \text{Var}_\lambda(X_1) \).

(b) Under what additional assumption on \( g \) and \( h \) does the usual method-of-moments estimator (MME) \( \hat{\lambda} \) exist with probability 1? Under this assumption, find a function \( V(\lambda) \) such that \( \sqrt{n}(\hat{\lambda} - \lambda) \overset{d}{\to} N(0, V(\lambda)) \) as \( n \to \infty \). If \( h(x) = g(-x) \) and \( \mu \neq 0 \), for what value of \( \lambda \) is \( V(\lambda) \) maximized?

(c) Assume that the maximum likelihood estimator (MLE) \( \hat{\lambda} \) exists with probability approaching 1 as \( n \to \infty \). Find a function \( W(\lambda) \) such that \( \sqrt{n}(\hat{\lambda} - \lambda) \overset{d}{\to} N(0, W(\lambda)) \) as \( n \to \infty \). (Your answer can involve an integral which you need not evaluate.) If \( h(x) = g(-x) \), for what value of \( \lambda \) is \( W(\lambda) \) maximized? [Hint: \( 1/f_\lambda(x) \) is convex in \( \lambda \).]

(d) it is well known that for regular (smooth) but non-exponential family models such as \( \{f_\lambda(x)\} \), the MLE \( \hat{\lambda} \) is more efficient than the MME \( \tilde{\lambda} \) in general, that is, \( W(\lambda) \leq V(\lambda) \). For the special case where \( g(x) = \frac{1}{2}(1 - x) \) and \( h(x) = g(-x) \) for \( x \in (-1, 1) \), find a value of \( \lambda \) such that \( W(\lambda) = V(\lambda) \).