Instructions

- There are six questions on this exam. Answer any five of them. If you attempt more than five, indicate below which five you want graded. If you do not provide such an indication, the first five will be graded.

- Each question is worth 20 points, although the questions vary in difficulty.

- Write your answers in the booklets provided. Use a new page for each question, and do not write on the back of the answer pages. The pages will be duplicated for grading, and if you write on the back, that material will not be graded and may adversely influence your grade. Label the pages at the top of each answer sheet with your ID number, the question number, and the sub-question numbers. If you fill up a booklet, ask for another.

- The examination is closed-book and closed-notes. Calculators and electronic devices are not allowed, and are not needed. The time limit is three hours.

- If you are using a pencil, please use only a number 2 lead pencil, to ensure adequate reproduction results.

Good luck!

Your name:__________________________________________________________

Problems to be graded (mark five):

1 __  2 __  3 __  4 __  5 __  6 __
Problem 1.
Let $X_1, X_2, \ldots, X_n$ be independent, identically distributed random variables having a uniform distribution $X_i \sim \mathcal{U}(0, \theta)$ for some unknown $\theta > 0$.

(a) For integer $k \geq 1$, derive an expression for the $k$th noncentral moment $\mu'_k = E[X_i^k]$.

(b) Derive an expression for $\text{Var}(X_i^k)$.

(c) Find a method of moments estimator $\tilde{\theta}^{(k)}$ based on the $k$th non central sample moment $m_k$ of the sample $(X_1, \ldots, X_n)$ and derive its asymptotic distribution.

(d) Show that for any specified value of $k$, there exists another method of moments estimator having an asymptotic distribution with smaller variance.

(e) Why would none of the method of moments estimators be preferred in this setting?
Problem 2.
Let $X_n \sim Binomial(n, p)$ with $0 < p < 1$. Set $Y_n = n - X_n$ and $q = 1 - p$.

(a) Find $E \left[ \frac{X_n}{Y_n} \right]$.

(b) Find $E \left[ \frac{X_n}{Y_n+1} \right]$. Show that it approaches $\frac{p}{q}$ as $n \to \infty$.

(c) Does there exist an unbiased estimator of $\frac{p}{q}$ based on $X_n$ and $Y_n$? Prove your answer rigorously.
Problem 3.
Let random vectors $\vec{Z}_1, \ldots, \vec{Z}_n$ be independent and identically distributed according to the bivariate normal distribution with

$$\vec{Z}_i = \begin{pmatrix} Z_{i1} \\ Z_{i2} \end{pmatrix} \sim \mathcal{N}_2 \left( \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} \theta & 0 \\ 0 & \theta \end{pmatrix} \right)$$

for $\theta > 0$. Let $Y_i = \sqrt{Z_{i1}^2 + Z_{i2}^2}$ be the length of random vector $\vec{Z}_i$. Then it can be shown (you do not have to prove this) that $Y_i$ has density function

$$f_{Y}(y) = \frac{y}{\theta} \exp\left\{-\frac{y^2}{2\theta}\right\} \mathbf{1}_{(0,\infty)}(y).$$

(a) Find $E[Y_i]$ and $Var(Y_i)$. [Hint: What is the distribution of $Y_i^2$?]

(b) Derive a level $\alpha$ uniformly most powerful test of $H_0 : \theta \leq \theta_0$ versus $\theta > \theta_0$. How would you find the critical value for the test?
Problem 4.
A researcher seeks to estimate a linear response function $g(x) = \beta_0 + \beta_1 x$ by taking measurements $y_i$ at specified input conditions $x_i$. The measurements are independent with expectations $E(y_i|x_i) = g(x_i)$ and equal variances. The researcher has a limited budget and can take exactly 100 measurements. Also, she is restricted to taking measurements at the following locations: $x = 0$ and $x = 1$. The experimental design problem is to determine the optimal (in terms of MSE) number of measurements, $n_0, n_1$ at the two points, with the constraint that $n_1 + n_0 = 100$.

(a) Prove that, if the goal is to estimate the slope $\beta_1$ of the linear function, then the optimal design is $n_0 = n_1 = 50$.

(b) Suppose now that the goal is to estimate $g(2)$, the response when $x = 2$, with minimum MSE. Derive the optimal design for this goal.
Problem 5.
Observe \((X_1, X_2, X_3) \sim \text{Multinomial}_3(n; p_1, p_2, p_1 + p_2)\), a trinomial distribution. (This is a full trinomial distribution, i.e., \(X_1 + X_2 + X_3 = n\).)

(a) What is the range of the parameter \(p_1\)?

(b) Find a minimal sufficient statistic \(T\). Is \(T\) complete? (Justify your answers.)

(c) If \(T\) is complete, find the UMVUE \(\tilde{p}_1\). If \(T\) is not complete, find a nontrivial ancillary statistic \(V\), find the conditional distribution \(T|V\), and find the conditional UMVUE \(\tilde{p}_1\) based on this conditional distribution.

(d) Find the MLE \(\hat{p}_1\) (state when it exists). Find the asymptotic distributions of \(\sqrt{n}(\hat{p}_1 - p_1)\) and \(\sqrt{n}(\tilde{p}_1 - p_1)\) as \(n \to \infty\).

(e) Because \(X_1 \sim \text{Binomial}(n, p_1)\), \(\tilde{p}_1 = \frac{X_1}{n}\) is an obvious unbiased estimator of \(p_1\). Find the ratio \(r\) of the asymptotic variances of \(\sqrt{n}(\hat{p}_1 - p_1)\) and \(\sqrt{n}(\tilde{p}_1 - p_1)\). How large can \(r\) be? How small? Which estimator \(\hat{p}_1\) or \(\tilde{p}_1\) is preferable?
Problem 6.
The number of calls coming into a switchboard is a Poisson process with rate $\lambda$ calls per 10 minutes. In other words, if $X$ is the number of calls in 10 minutes, then $X \sim \text{Poisson}(\lambda)$. The problem is to find the best unbiased estimate for $\theta = \mathbb{P}(0 \text{ calls in the next 20 minutes})$ based on the single data point $X$ (the number of calls in the previous 10 minutes).

(a) State the distribution of $Y$, the number of calls in 20 minutes.

(b) Express $\theta$ in terms of $\lambda$.

(c) Is $X$ a complete sufficient statistic for $\lambda$? Why or why not?

(d) Find an unbiased estimate of $\theta$ based on $X$. [You may find it helpful to use the Taylor expansion $e^{-\lambda} = \sum_{x=0}^{\infty} \frac{(-1)^x \lambda^x}{x!}$.]

(e) Explain why the estimate you found in (d) is the best unbiased estimate.

(f) Comment briefly whether you would choose to use the best unbiased estimate of $\theta$. 

Solution 1.
Let \( X_1, X_2, \ldots, X_n \) be independent, identically distributed random variables having a uniform distribution \( X_i \sim \mathcal{U}(0, \theta) \) for some unknown \( \theta > 0 \).

(a) For integer \( k \geq 1 \), derive an expression for the \( k \)th noncentral moment \( \mu'_k = E[X_i^k] \).
Answer:
\[
\mu'_k = \int_0^\theta x^k \frac{1}{\theta} \, dx = \left[ \frac{x^{k+1}}{(k+1)} \right]_0^\theta = \frac{\theta^k}{(k+1)}
\]

(b) Derive an expression for \( \text{Var}(X_i^k) \).
Answer: Using the above expression that holds for arbitrary \( k \)
\[
\text{Var}(X_i^k) = E[X_i^{2k}] - E^2[X_i^k] = \frac{\theta^{2k}}{(2k+1)} - \frac{\theta^{2k}}{(k+1)^2} = \theta^{2k} \left( \frac{k^2}{(2k+1)(k+1)^2} \right)
\]

(c) Find a method of moments estimator \( \hat{\theta}^{(k)} \) based on the \( k \)th non-central sample moment \( m_k \) of the sample \( (X_1, \ldots, X_n) \) and derive its asymptotic distribution.
Answer: We find MME
\[
\hat{\theta}^{(k)} = \left( (k+1)m_k \right)^\frac{1}{k}
\]
Now, by the CLT, we have that
\[
\sqrt{n} \left( m_k - E[X_i^k] \right) \rightarrow_d \mathcal{N}(0, \text{Var}(X_i^k)) \Rightarrow \sqrt{n} \left( m_k - \frac{\theta^k}{(k+1)} \right) \rightarrow_d \mathcal{N} \left( 0, \theta^{2k} \left( \frac{k^2}{(2k+1)(k+1)^2} \right) \right),
\]
so using the delta method with \( g(u) = ((k+1)u)^\frac{1}{k} \) and \( g'(u) = \frac{(k+1)^{1/k}}{k} u^{\frac{1}{k}-1} \)
\[
\sqrt{n} \left( \hat{\theta}^{(k)} - \theta \right) \rightarrow_d \mathcal{N} \left( 0, \frac{\theta^2}{(2k+1)} \right)
\]

(d) Show that for any specified value of \( k \), there exists another method of moments estimator having an asymptotic distribution with smaller variance.
Answer: By inspection, the asymptotic variance is decreasing in \( k \), so the MME for any \( k' > k \) would have smaller asymptotic variance.

(e) Why would none of the method of moments estimators be preferred in this setting?
Answer: The sufficient statistic in this probability model is the sample maximum, which is asymptotically \( n \)-consistent (but exponentially distributed), and hence more efficient than the \( \sqrt{n} \)-consistent MME \( \hat{\theta}^{(k)} \). Note that the \( L_\infty \) norm is the sample maximum, so the better behavior of higher \( k \) agrees with the MLE, though we cannot use the limiting asymptotic distribution from the MMEs. (The students do not need to derive anything here, if they can state facts from memory.)
Solution 2.
Let \( X_n \sim \text{Binomial}(n, p) \) with \( 0 < p < 1 \). Set \( Y_n = n - X_n \) and \( q = 1 - p \).

(a) Find \( E \left[ \frac{X_n}{Y_n} \right] \).
Answer: \( E \left[ \frac{X_n}{Y_n} \right] = \infty \) because \( P[Y_n = 0] = p^n \), so \( E \left[ \frac{X_n}{Y_n} \right] = \infty \).

(b) Find \( E \left[ \frac{X_n}{Y_n + 1} \right] \). Show that it approaches \( \frac{p}{q} \) as \( n \to \infty \).
Answer:
\[
E \left[ \frac{X_n}{Y_n + 1} \right] = \sum_{x=0}^{n} \frac{x}{n-x+1} \frac{n!}{x!(n-x)!} p^x q^{n-x} = \sum_{x=1}^{n} \frac{n!}{(x-1)!(n-x+1)!} p^x q^{n-x} \\
= \sum_{k=0}^{n-1} \frac{n!}{k!(n-k)!} p^{k+1} q^{n-k-1} = \frac{p}{q} \sum_{k=0}^{n-1} \frac{n!}{k!(n-k)!} q^k = \frac{p}{q} \sum_{k=0}^{n-1} \frac{n!}{k!(n-k)!} q^k = \frac{p}{q} (1 - q^n) \to \frac{p}{q} \text{ as } n \to \infty
\]

(c) Does there exist an unbiased estimator of \( \frac{p}{q} \) based on \( X_n \) and \( Y_n \)? Prove your answer rigorously.
Answer: No.
Suppose that \( E[g(X_n)] = \frac{p}{q} \) for all \( 0 < p < 1 \). Thus
\[
\frac{p}{q} = \sum_{x=0}^{n} g(x) \frac{n!}{x!(n-x)!} p^x q^{n-x}
\]
so
\[
p = \sum_{x=0}^{n} g(x) \frac{n!}{x!(n-x)!} p^x q^{n-x+1} \to g(0) \text{ as } p \to 0,
\]
so \( g(0) = 0 \). Then let \( p \to 1 \) (equivalently \( q \to 0 \)) to obtain the contradiction \( 1 = 0 \).
Solution 3.
Let random vectors \( \vec{Z}_1, \ldots, \vec{Z}_n \) be independent and identically distributed according to the bivariate normal distribution with
\[
\vec{Z}_i = \begin{pmatrix} Z_{i1} \\ Z_{i2} \end{pmatrix} \sim \mathcal{N}_2 \left( \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} \theta & 0 \\ 0 & \theta \end{pmatrix} \right)
\]
for \( \theta > 0 \). Let \( Y_i = \sqrt{Z_{i1}^2 + Z_{i2}^2} \) be the length of random vector \( \vec{Z}_i \). Then it can be shown (you do not have to prove this) that \( Y_i \) has density function
\[
f_{Y}(y) = \frac{y}{\theta} \exp \left\{ -\frac{y^2}{2\theta} \right\} 1_{(0, \infty)}(y).
\]

(a) Find \( E[Y_i] \) and \( Var(Y_i) \). [Hint: What is the distribution of \( Y_i^2 \)?]

**Answer:** By straightforward integration, we find the mean of this Rayleigh distribution is
\[
E[Y_i] = \sqrt{\frac{\pi}{\theta}}.
\]

By appealing to the chi square distribution of the sum of squared iid normal random variables, we find
\( E[Y_i^2] = 2\theta \), and thus \( Var(Y_i) = \frac{4 - \pi^2}{2} \theta \).

(b) Derive a level \( \alpha \) uniformly most powerful test of \( H_0 : \theta \leq \theta_0 \) versus \( \theta > \theta_0 \). How would you find the critical value for the test?

**Answer:** The likelihood is given by
\[
L(\theta | \vec{Y}) = \frac{1}{\theta^n} \prod_{i=1}^n Y_i \exp \left\{ -\frac{\sum_{i=1}^n Y_i^2}{2\theta} \right\} 1_{(0, \infty)}(Y_{(1)})
\]
and for \( \theta_1 > \theta_0 \) the log likelihood ratio
\[
\log \Lambda(\theta_1, \theta_0) = n \log \left( \frac{\theta_0}{\theta_1} \right) - \frac{1}{2} \sum_{i=1}^n Y_i^2 \left( \frac{\theta_0 - \theta_1}{\theta_1 \theta_0} \right)
\]
is increasing in \( T = \sum_{i=1}^n Y_i^2 \), so we have monotone likelihood ratio in \( T \), and by the Karlin-Rubin theorem the UMP-\( \alpha \) test will reject for \( T > c \). Now we know that \( Y_i^2 \sim \theta \chi^2_2 \), and the \( Y_i \)'s are independent, so \( T \sim \theta \chi^2_{2n} \) and we can find the critical value based on
\[
\text{reject } H_0 \iff \frac{1}{\theta_0} \sum_{i=1}^n Y_i^2 > \chi^2_{2n, 1-\alpha},
\]
the \( 1 - \alpha \) critical value of the chi square distribution having \( 2n \) degrees of freedom.

Note that we could also use the result that for a one parameter exponential family of the form \( h(T) e^{w(\theta)T} \) for with \( w(\theta) = -\frac{1}{\theta} \) increasing in \( \theta \), we have monotone likelihood ratio in \( T \) and our UMP-\( \alpha \) test will reject for \( T > c \) as above.
Solution 4.
A researcher seeks to estimate a linear response function \( g(x) = \beta_0 + \beta_1 x \) by taking measurements \( y_i \) at specified input conditions \( x_i \). The measurements are independent with expectations \( E(y_i|x_i) = g(x_i) \) and equal variances. The researcher has a limited budget and can take exactly 100 measurements. Also, she is restricted to taking measurements at the following locations: \( x = 0 \) and \( x = 1 \). The experimental design problem is to determine the optimal (in terms of MSE) number of measurements, \( n_0, n_1 \) at the two points, with the constraint that \( n_1 + n_0 = 100 \).

(a) Prove that, if the goal is to estimate the slope \( \beta_1 \) of the linear function, then the optimal design is \( n_0 = n_1 = 50 \).

**Answer:** The best linear unbiased estimator of \( \beta_1 \) in this homoscedastic linear model will be the ordinary least squares estimator

\[
\hat{\beta}_1 = (X^T X)^{-1} X^T \bar{Y}
\]

which will have moments

\[
\hat{\beta}_1 \sim \left( \beta_1 \sigma^2 (X^T X)^{-1} \right).
\]

The MSE will be minimized by minimizing the variance. For the design matrix \( X \) having \( n_0 \) rows of \((1 \ 0)\) and \( n_1 \) rows of \((1 \ 1)\), we will have

\[
(X^T X) = \begin{pmatrix} n_0 + n_1 & n_1 \\ n_1 & n_1 \end{pmatrix} \quad \iff \quad (X^T X)^{-1} = \begin{pmatrix} \frac{1}{n_0} & -\frac{1}{n_0} \\ -\frac{1}{n_0} & \frac{1}{n_0} + \frac{1}{n_1} \end{pmatrix}.
\]

We thus want to minimize \( \text{Var}(\hat{\beta}_1) = \sigma^2 \left( \frac{1}{n_0} + \frac{1}{n_1} \right) \) subject to \( n_0 + n_1 = n = 100 \), so

\[
\frac{d}{dn_0} \left( \frac{1}{n_0} + \frac{1}{n - n_0} \right) = 0 \quad \iff \quad \left( \frac{-1}{n_0^2} + \frac{1}{(n-n_0)^2} \right) = 0 \quad \iff \quad n_0 = \frac{n}{2} = 50.
\]

(b) Suppose now that the goal is to estimate \( g(2) \), the response when \( x = 2 \), with minimum MSE. Derive the optimal design for this goal.

**Answer:** We now want to estimate \( \bar{a}^T \hat{\beta}_1 \) for \( \bar{a}^T = (1 \ 2) \). We thus have moments for our least squares estimate given by

\[
\bar{a}^T \hat{\beta}_1 \sim \left( \bar{a}^T \hat{\beta}, \sigma^2 \bar{a}^T (X^T X)^{-1} \bar{a} \right).
\]

We thus find

\[
\bar{a}^T (X^T X)^{-1} \bar{a} = \begin{pmatrix} 1 & 2 \end{pmatrix} \begin{pmatrix} \frac{1}{n_0} & -\frac{1}{n_0} \\ -\frac{1}{n_0} & \frac{1}{n_0} + \frac{1}{n_1} \end{pmatrix} \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \frac{1}{n_0} - \frac{2}{n_0} - \frac{2}{n_0} + \frac{4}{n_0} + \frac{4}{n_1} = \frac{1}{n_0} + \frac{4}{n_1}.
\]

We thus want to minimize \( \sigma^2 \left( \frac{1}{n_0} + \frac{4}{n_1} \right) \) subject to \( n_0 + n_1 = n = 100 \), so

\[
\frac{d}{dn_0} \left( \frac{1}{n_0} + \frac{4}{n - n_0} \right) = 0 \quad \iff \quad \left( \frac{-1}{n_0^2} + \frac{4}{(n-n_0)^2} \right) = 0 \quad \iff \quad 3n_0^2 + 2nn_0 - n^2 = 0
\]

which upon factoring yields

\[
(3n_0 - n)(n_0 + n) = 0
\]

and taking the positive root, we find \( n_0 = \frac{n}{3} = \frac{100}{3} \). Because we have to take an integer sample size, we can consider both \( n_0 = 33 \) and \( n_0 = 34 \) to find the smaller variance is given by \( n_0 = 33 \).
Solution 5.

Observe \((X_1, X_2, X_3) \sim \text{Multinomial}_3(n; p_1, p_2, p_1 + p_2)\), a trinomial distribution. (This is a full trinomial distribution, i.e., \(X_1 + X_2 + X_3 = n\).)

(a) What is the range of the parameter \(p_1\)?
Answer: \(p_1 + p_2 + (p_1 + p_2) = 1\), so \(p_1 + p_2 = \frac{1}{2}\), hence \(0 < p_1 < \frac{1}{2}\). Thus \((X_1, X_2, X_3) \sim \text{Mult}_3(n; p_1, \frac{1}{2} - p_1, \frac{1}{2})\).

(b) Find a minimal sufficient statistic \(T\). Is \(T\) complete? (Justify your answers.)
Answer:
\[
f_{p_1}(x_1, x_2, x_3) = \frac{n!}{x_1!x_2!(n - x_1 - x_2)!}p_1^{x_1}(1 - p_1)^{x_2}(1 - x_1 - x_2) \equiv e^{x_1 \theta_1 + x_2 \theta_2} h(x_1, x_2),
\]
where \(\theta_1 = \log(p_1)\) and \(\theta_2 = \log\left(\frac{1}{2} - p_1\right)\). Because the range of \((\theta_1, \theta_2)\) affinity spans \(\mathbb{R}^2\) [verify], \(T \equiv (X_1, X_2)\) is minimal sufficient. However \(T\) is not complete because it contains the nontrivial ancillary statistic \(V \equiv X_1 + X_2\).

(c) If \(T\) is complete, find the UMVUE \(\tilde{p}_1\). If \(T\) is not complete, find a nontrivial ancillary statistic \(V\), find the conditional distribution \(T \mid V\), and find the conditional UMVUE \(\tilde{p}_1\) based on this conditional distribution.
Answer: \(T \mid V \equiv (X_1, V - X_1) \mid V \sim \text{Binomial}(V, \frac{p_1}{p_1 + p_2}) = \text{Binomial}(V, 2p_1)\). Therefore the conditional UMVUE is \(\tilde{p}_1 = \frac{X_1}{V}\) (provided \(V > 0\)).

(d) Find the MLE \(\hat{p}_1\) (state when it exists). Find the asymptotic distributions of \(\sqrt{n}(\hat{p}_1 - p_1)\) and \(\sqrt{n}(\tilde{p}_1 - p_1)\) as \(n \to \infty\).
Answer: From \(f_{p_1}(x_1, x_2, x_3)\) above, \(\hat{p}_1 = \frac{X_1}{n} = \tilde{p}_1\) provided that \(V > 0\). Since the ancillary \(V \sim \text{Binomial}(n, \frac{1}{2})\), note that \(Pr[V > 0] = 1 - \left(\frac{1}{2}\right)^n \to 1\) as \(n \to \infty\). Because this trinomial model is regular,
\[
\sqrt{n}(\hat{p}_1 - p_1) \to d \mathcal{N}(0, \frac{1}{I(p_1)}),
\]
where
\[
I(p_1) = -\frac{1}{n} E_{p_1} \left[ \frac{\partial^2 f_{p_1}(X_1, X_2, X_3)}{\partial p_1^2} \right] = \frac{1}{n} E_{p_1} \left[ \frac{X_1}{p_1^2} + \frac{X_2}{(\frac{1}{2} - p_1)^2} \right] \equiv \frac{1}{p_1(1 - 2p_1)}.
\]

(e) Because \(X_1 \sim \text{Binomial}(n, p_1)\), \(\hat{p}_1 \equiv \frac{X_1}{n}\) is an obvious unbiased estimator of \(p_1\). Find the ratio \(r\) of the asymptotic variances of \(\sqrt{n}(\hat{p}_1 - p_1)\) and \(\sqrt{n}(\tilde{p}_1 - p_1)\). How large can \(r\) be? How small? Which estimator \(\hat{p}_1\) or \(\tilde{p}_1\) is preferable?
Answer: From the CLT, \(\sqrt{n}(\hat{p}_1 - p_1) \to d \mathcal{N}(0, p_1(1 - p_1))\), so \(r = \frac{1 - 2p_1}{1 - p_1}\). Thus \(r\) decreases from 1 to 0 as \(p_1\) goes from 0 to \(\frac{1}{2}\). Clearly \(\hat{p}_1\) is preferable.
Solution 6.
The number of calls coming into a switchboard is a Poisson process with rate \( \lambda \) calls per 10 minutes. In other words, if \( X \) is the number of calls in 10 minutes, then \( X \sim \text{Poisson}(\lambda) \). The problem is to find the best unbiased estimate for \( \theta = P(0 \text{ calls in the next 20 minutes}) \) based on the single data point \( X \) (the number of calls in the previous 10 minutes).

(a) State the distribution of \( Y \), the number of calls in 20 minutes.
Answer: \( Y \sim \text{Poisson}(2\lambda) \)

(b) Express \( \theta \) in terms of \( \lambda \).
Answer: \( \theta = P(0 \text{ calls in 20 minutes}) = P(Y = 0) = e^{-2\lambda(2\lambda)^0}/0! = e^{-2\lambda} \)

(c) Is \( X \) a complete sufficient statistic for \( \lambda \)? Why or why not?
Answer: \( X \) is a sample from an exponential family of distributions: \( f(X|\lambda) = \frac{e^{-\lambda}x^\lambda}{x!}e^{x\log \lambda} \). Therefore \( X \) is a complete sufficient statistic.

(d) Find an unbiased estimate of \( \theta \) based on \( X \). [You may find it helpful to use the Taylor expansion \( e^{-\lambda} = \sum_{x=0}^{\infty} \frac{(-1)^x\lambda^x}{x!} \).]
Answer: Since \( X \) is a complete sufficient statistic, it suffices to find a function \( g() \) such that \( E[g(X)] = \theta \).

\[
E[g(X)] = \sum_{x=0}^{\infty} g(x)\frac{e^{-\lambda}\lambda^x}{x!} = e^{-2\lambda}
\]

\[
\sum_{x=0}^{\infty} g(x)\frac{\lambda^x}{x!} = e^{-\lambda} = \sum_{x=0}^{\infty} \frac{(-1)^x\lambda^x}{x!}
\]

Equating coefficients in the infinite sum, \( g(x) = (-1)^x \). In other words, we estimate \( \hat{\theta} = 1 \) if \( X = 0, 2, 4, \ldots \) and \( \hat{\theta} = -1 \) if \( X = 1, 3, 5, \ldots \).

(e) Explain why the estimate you found in (d) is the best unbiased estimate.
Answer: Since \( X \) is complete, this will be the only unbiased estimate that is a function of \( X \) and therefore the best.

(f) Comment briefly whether you would choose to use the best unbiased estimate of \( \theta \).
Answer: This is a terrible estimate, leading one to estimate that the probability of 0 calls in 20 minutes is either 1 or -1 (which is not even a valid probability)