5.0 Functions and limits

Let X and Y be two sets

A function from X to Y is a rule that assigns to each element of X a unique element in Y

Remarks

① Denote the function by "f"

② Call X the domain of f

③ Call Y the range of f

④ Refer to y as the value of f at x

⑤ Denote

$$y = f(x) \qquad \forall x \in X$$

⑥ Call x the argument of f and the independent variable

⑦ Call y the dependent variable

Examples of functions

$f(x) = y = x^2$ $\quad X = \mathbb{R}$, $Y = \{x : x \geq 0\}$

$f(x) = y = \beta_0 + \beta_1 x$ $\quad X = \mathbb{R}$, $Y = \mathbb{R}$ if $\beta_1 \neq 0$

$f(x) = y = ax^2 + bx + c$ $\quad a \neq 0$

$f(x) = a x^b$

Inverse functions

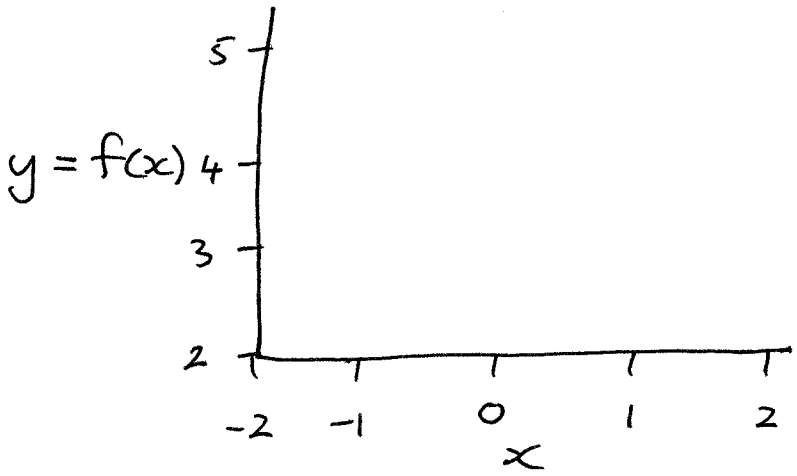
If $f(x) = y$ then the function

$g(y) = x$ $\quad \forall$ x and y

is called the inverse function of $f(x)$.

It is denoted by $f^{-1}(y)$

5.1 Graphical representation of functions



$f(x) = 2 + 0.5\,x^2 \qquad x \in \mathbb{R}$

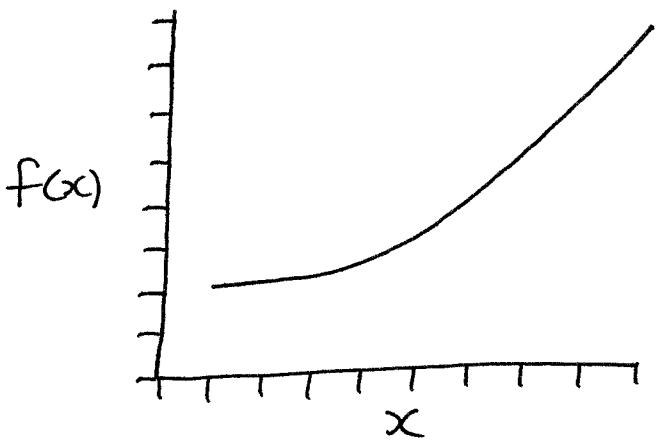
So $Y =$

5.2 Slopes of functions

- generalize the idea of the slope of a line
- slope: the change in $f(\cdot)$ as x changes to $x+\Delta x$ divided by Δx.

  that is, the average rate of change in $f(\cdot)$ from x to $x+\Delta x$

Geometrically



5.3 Derivatives of functions

Idea: The derivative is the value of the slope the is produced as Δx becomes progressively smaller

The derivative of $f(x)$ at x is

$$\lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

Denote the derivative by

$$\frac{dy}{dx}$$

5.4 Derivatives of power and exponential functions

What is the derivative of $f(x) = x$?

What is the derivative of $f(x) = x^2$?

What is the derivative of $f(x) = x^n$ $n > 0$

Can show the derivative of e^{ax} is

$$a e^{x}$$

and so the derivative of

$$f(x) = b^{x}$$

is

$$(\log_e b)\, b^{x}$$

5.5 The calculation of derivatives

Some rules

① $$\frac{d(af(x))}{dx} = a\frac{df(x)}{dx}$$

② $$\frac{d(f(x)+g(x))}{dx} = \frac{df(x)}{dx} + \frac{dg(x)}{dx}$$

③ $$\frac{d(f(x)\cdot g(x))}{dx} = f(x)\cdot\frac{dg(x)}{dx} + g(x)\cdot\frac{df(x)}{dx}$$

This is called the product rule

④ $$\frac{d(f(g(x)))}{dx} = \frac{df(g(x))}{dx}\cdot\frac{dg(x)}{dx}$$

This is called the chain rule

Examples

What is the derivative of $h(x) = (5x-3)(x+1)$?

$$\frac{dh(x)}{dx} = \frac{d(5x-3)}{dx} \cdot (x+1) + \frac{d(x+1)}{dx} \cdot (5x-3)$$

$$= 5(x+1) + (5x-3) = 10x + 2$$

What is the derivative of $h(x) = (x^2+1)^5$

$$\frac{df(x)}{dx} = 5x^4$$

$$\frac{dh(x)}{dx} = \frac{df}{dx}\left(g(x)\right) \cdot \frac{dg(x)}{dx}$$

$$= 5(x^2+1)^4 \cdot 2x$$

$$= 10x\,(x^2+1)^4$$