CSSS 505

Sequence and Series Summary Formulas

1. If a sequence $\{a_n\}$ has a limit L, that is, $\lim_{n\to\infty} a_n = L$, then the sequence is said to converge to L. If there is no limit, the series diverges. If the sequence $\{a_n\}$ converges, then its limit is unique. Keep in mind that $\lim_{n\to\infty} \frac{\ln n}{n} = 0$; $\lim_{n\to\infty} x^{\left(\frac{1}{n}\right)} = 1$; $\lim_{n\to\infty} \sqrt[n]{n} = 1$; $\lim_{n\to\infty} \frac{x^n}{n!} = 0$. These limits are useful and arise frequently.
2. The harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges; the geometric series $\sum_{n=0}^{\infty} ar^n$ converges to $\frac{a}{1-r}$ if $|r| < 1$ and diverges if $|r| \geq 1$ and $a \neq 0$.
3. The p-series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges if $p > 1$ and diverges if $p \leq 1$.
4. Limit Comparison Test: Let $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ be a series of nonnegative terms, with $a_n \neq 0$ for all sufficiently large n, and suppose that $\lim_{n\to\infty} \frac{b_n}{a_n} = c > 0$. Then the two series either both converge or both diverge.
5. Alternating Series: Let $\sum_{n=1}^{\infty} a_n$ be a series such that
   - i) the series is alternating
   - ii) $\left| a_{n+1} \right| \leq \left| a_n \right|$ for all n, and
   - iii) $\lim_{n\to\infty} a_n = 0$

   Then the series converges.
6. A series $\sum a_n$ is absolutely convergent if the series $\sum |a_n|$ converges. If $\sum a_n$ converges, but $\sum |a_n|$ does not converge, then the series is conditionally convergent. Keep in mind that if $\sum_{n=1}^{\infty} |a_n|$ converges, then $\sum_{n=1}^{\infty} a_n$ converges.

7. Comparison Test: If $0 \le a_n \le b_n$ for all sufficiently large n, and $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n$ converges. If $\sum_{n=1}^{\infty} a_n$ diverges, then $\sum_{n=1}^{\infty} b_n$ diverges.

8. Integral Test: If $f(x)$ is a positive, continuous, and decreasing function on $[1,\infty)$ and let $a_n = f(n)$. Then the series $\sum_{n=1}^{\infty} a_n$ will converge if the improper integral $\int_1^{\infty} f(x)\,dx$ converges. If the improper integral $\int_1^{\infty} f(x)\,dx$ diverges, then the infinite series $\sum_{n=1}^{\infty} a_n$ diverges.

9. Ratio Test: Let $\sum a_n$ be a series with nonzero terms.

   i) If $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| < 1$, then the series converges absolutely.

   ii) If $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| > 1$, then the series is divergent.

   iii) If $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = 1$, then the test is inconclusive (and another test must be used).

10. Power Series: A power series is a series of the form

$$\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + \ldots + c_n x^n + \ldots \quad \text{or}$$

$$\sum_{n=0}^{\infty} c_n (x-a)^n = c_0 + c_1 (x-a) + c_2 (x-a)^2 + \ldots + c_n (x-a)^n + \ldots$$

in which the center a and the coefficients $c_0, c_1, c_2, \ldots, c_n, \ldots$ are constants. The set of all numbers x for which the power series converges is called the interval of convergence.

11. Taylor Series: Let f be a function with derivatives of all orders throughout some interval containing a as an interior point. Then the Taylor series generated by f at a is

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \ldots + \frac{f^{(n)}(a)}{n!}(x-a)^n + \ldots$$

The remaining terms after the term containing the nth derivative can be expressed as a remainder to Taylor's Theorem:

$$f(x) = f(a) + \sum_{1}^{n} f^{(n)}(a)(x-a)^n + R_n(x) \quad \text{where} \quad R_n(x) = \frac{1}{n!}\int_a^x (x-t)^n f^{(n+1)}(t)\,dt$$

Lagrange's form of the remainder: $R_n x = \dfrac{f^{(n+1)}(c)(x-a)^{n+1}}{(n+1)!}$, where $a < c < x$. The series will converge for all values of x for which the remainder goes to zero.

12. Frequently Used Series

$$\frac{1}{1-x} = 1 + x + x^2 + \ldots + x^n + \ldots = \sum_{n=0}^{\infty} x^n \ , \ |x| < 1$$

$$\frac{1}{1+x} = 1 - x + x^2 - \ldots + (-x)^n + \ldots = \sum_{n=0}^{\infty} (-1)^n x^n \ , \ |x| < 1$$

$$e^x = 1 + x + \frac{x^2}{2!} + \ldots + \frac{x^n}{n!} + \ldots = \sum_{n=0}^{\infty} \frac{x^n}{n!}, \ |x| < \infty$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots + (-1)^n \frac{x^{2n+1}}{(2n+1)!} + \ldots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}, \ |x| < \infty$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots + (-1)^n \frac{x^{2n}}{(2n)!} + \ldots = \sum_{n=0}^{\infty} \frac{(-1)x^{2n}}{(2n)!}, \ |x| < \infty$$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \ldots + (-1)^{n-1} \frac{x^n}{n} + \ldots = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} x^n}{n}, \ -1 < x \le 1$$

$$Arc \tan x = x - \frac{x^3}{3} + \frac{x^5}{5} - \ldots + (-1)^n \frac{x^{2n+1}}{2n+1} + \ldots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1}, \ |x| \le 1$$