HW6: Discrete Random Variables (2) – Solutions

Problem 1. Four buses carrying 148 students from the same school arrive at a football stadium. The buses carry, respectively, 40, 33, 25, and 50 students. One of the students is randomly selected. Let $X$ denote the number of students that were on the bus carrying the randomly selected student. One of the 4 bus drivers is also randomly selected. Let $Y$ denote the number of students on her bus.

(a) Compute $E[X]$ and $E[Y]$.

Answer. Both $X$ and $Y$ can take on the same values: 40, 33, 25, and 50.

The probability mass function $p_X$ of $X$ is given by:

- $p_X(40) = 40/148$
- $p_X(33) = 33/148$
- $p_X(25) = 25/148$
- $p_X(50) = 50/148$

Since the bus driver is equally likely to drive any of the 4 buses, the probability mass function $p_Y$ of $Y$ is $p_Y(40) = p_Y(33) = p_Y(25) = p_Y(50) = 1/4$.

The respective expectations are given by:

\[
E[X] = \sum_{x \in \{40, 33, 25, 50\}} xp_X(x)
= 40 \cdot p_X(40) + 33 \cdot p_X(33) + 25 \cdot p_X(25) + 50 \cdot p_X(50)
\approx 39.28
\]

and

\[
E[Y] = \sum_{x \in \{40, 33, 25, 50\}} xp_Y(x)
= \frac{1}{4} \cdot (40 + 33 + 25 + 50)
= 37
\]

(b) Find $\text{Var}(X)$ and $\text{Var}(Y)$ for $X$ and $Y$ respectively.
Answer. We have the following:

\[ E[X^2] = \sum_{x \in \{40, 33, 25, 50\}} x^2 p_X(x) \]

\[ = 40^2 \cdot p_X(40) + 33^2 \cdot p_X(33) + 25^2 \cdot p_X(25) + 50^2 \cdot p_X(50) \]

\[ = 1625.4 \]

and

\[ E[Y^2] = \sum_{x \in \{40, 33, 25, 50\}} x^2 p_Y(x) \]

\[ = \frac{1}{4} \cdot (40^2 + 33^2 + 25^2 + 50^2) \]

\[ = 1453.5 \]

Therefore, the variances of \( X \) and \( Y \) are given by:

\[ \text{Var}(X) = E[X^2] - (E[X])^2 \approx 82.18 \]

and

\[ \text{Var}(Y) = E[Y^2] - (E[Y])^2 = 84.5 \]

Problem 2. An insurance company writes a policy to the effect that an amount of money \( A \) must be paid if some event \( E \) occurs within a year. If the company estimates that \( E \) will occur within a year with probability \( p \), what should it charge the customer in order that its expected profit will be 10 percent of \( A \)?

Answer. Let \( C \) be the amount that the insurance company charges the customer. Let \( X \) denote its profit. The company’s profit is \( C - A \) if event \( E \) occurs and it will gain \( C \) if \( E \) does not occur. The expected profit would thus be:

\[ E[X] = p(C - A) + (1 - p)C = -pA + C \]

Amount \( C \) is found by solving: 0.1 \( \cdot \) \( A \) = \( C \) - \( p \) \( A \). Therefore \( C = A(0.1 + p) \)

Problem 3. Each night different meteorologists give us the probability that it will rain the next day. To judge how well these people predict, we will score each of them as follows: If a meteorologist says that it will rain with probability \( p \), then he or she will receive a score of

\[ 1 - (1 - p)^2 \text{ if it does rain} \]

\[ 1 - p^2 \text{ if it does not rain} \]

We will then keep track of scores over a certain time span and conclude that the meteorologist with the highest average score is the best predictor of weather. Suppose now that a given meteorologist is aware of our scoring mechanism and wants to maximize his or her expected score. If this person truly believes that it
will rain tomorrow with probability $p^*$, what value of $p$ should he or she assert so as to maximize the expected score?

**Hint.** Differentiate the expected score and find the value of $p$ that sets the derivative to 0.

**Answer.** If that meteorologist believes that it will rain tomorrow with probability $p^*$, her expected score denoted $S(p)$ will be:

$$S(p) = (1 - (1 - p)^2) p^* + (1 - p^2) (1 - p^*)$$

To find the value of $p$ that maximize $S(p)$, let us calculate the derivative $S'(p)$:

$$S'(p) = (-2 \cdot (-1) \cdot (1 - p)) p^* + (-2p)(1 - p^*)$$
$$= 2p^*(1 - p) - 2p(1 - p^*)$$
$$= 2p^* - 2p$$

Setting $S'(p)$ to zero, we get $p = p^*$. Thus the meteorologist has an incentive to predict honestly.

**Problem 4.** Let $X$ be a real-valued random variable. If $E[X] = 1$ and $\text{Var}(X) = 5$, find

(a) $E[(2 + X)^2]$;

**Answer.**

$$E[(2 + X)^2] = E[4 + 4X + X^2]$$
$$= 4 + 4E[X] + E[X^2], \text{ by linearity of the expectation}$$

The shortcut formula for the variance provides $E[X^2] = \text{Var}(X) + (E[X])^2$. Therefore,

$$E[(2 + X)^2] = 4 + 4 \cdot 1 + (5 + 1^2) = 14$$

(b) $\text{Var}(4 + 3X)$.

**Answer.**

$$\text{Var}(4 + 3X) = 3^2 \text{Var}(X) = 9 \cdot 5 = 45$$