Problem 1. A gambling book recommends the following “winning strategy” for the game of roulette: Bet $1 on red. If red appears (which has probability $18/38$), then take the $1 profit and quit. If red does not appear and you lose this bet, which has probability $20/38$ of occurring, make additional $1 bets on red on each of the next two spins of the roulette wheel and then quit. Let $X$ denote your winnings when you quit.

(a) Find $P(X > 0)$.

Answer. Let $R$ denote the outcome that the ball lands on red and $R^c$ if not. The proposed strategy consists of the following possible sequences: $R$, $R^c RR$, $R^c RR^c$, $R^c R R$ and $R^c R^c R^c$. The winnings associated with those events are respectively $+1$, $+1$, $-1$, $-1$, $-3$. The corresponding probabilities are:

- $P(R) = 18/38$
- $P(R^c RR) = (20/38) \cdot (18/38)^2$
- $P(R^c RR^c) = (18/38) \cdot (20/38)^2$
- $P(R^c R R) = (20/38)^2 \cdot (18/38)$
- $P(R^c R^c R) = (20/38)^3$

Thus, the desired probability is:

$$P(X > 0) = P(R) + P(R^c RR) \approx 0.5918.$$ 

(b) Find $E[X]$ the expected value of $X$. Is the strategy indeed a “winning strategy”?

Answer.

$$E[X] = \sum_{x \in \{-1, 1, -3\}} x p_X(x)$$

$$= -1 \cdot p_X(-1) + 1 \cdot p_X(1) + (-3)p_X(-3)$$

$$= -1 \cdot (P(R^c RR^c) + P(R^c RR)) + 1 \cdot (P(R) + P(R^c RR)) + (-3)p_X(R^c R^c)$$

$$\approx -0.108$$

So it is not a winning strategy.
Problem 2. A box contains 5 red and 5 blue marbles. Two marbles are withdrawn randomly. If they are the same color, then you win $1.10; if they are different colors, then you win -$1.00. (That is, you lose $1.00). Calculate

(a) the expected value of the amount you win:

Answer. Let $W$ be the amount of winnings. The probability of winning is the probability of drawing 2 balls of the same color, which is $\binom{5}{2} / \binom{10}{2}$, which is also equal to $(10 \cdot 4)/(10 \cdot 9) = 4/9$. Therefore, $E[W] = -1 \cdot (1 - 4/9) + 1.10 \cdot 4/9 = -1/15$. 

(b) the variance of the amount you win.

Answer. We will use the shortcut formula (König-Huygens) : $\text{Var}(W) = E[W^2] - (E[W])^2$.

$E[W^2] = (-1)^2 \cdot (1 - 4/9) + 1.10^2 \cdot 4/9$.

Hence, we have :

$\text{Var}(W) = (-1)^2 \cdot (1 - 4/9) + 1.10^2 \cdot 4/9 - (-1/15)^2 \approx 1.089.$

Problem 3. Blue eyes is a recessive trait. When two people who have brown eyes but have a parent with blue eyes then their kids are blue-eyed with probability 1/4 and brown-eyed with probability 3/4. Alice and Bob both have brown eyes but one of their parents have blue eyes. If Bob and Alice have four kids, what is the probability that

(a) three of them have blue eyes?

Answer. Let $X$ be the number of Alice and Bob’s kids that have blue eyes. Then $X$ obeys a binomial distribution law $Bin(n, p)$ with parameters $n = 4$ and $p = 0.25$. Therefore,

$p_X(3) = \binom{n}{3} p^3 (1 - p)^{n-3} = \binom{4}{3} 0.25^3 \cdot 0.75^1 \approx 0.047.$

(b) three of them have brown eyes?

Answer. If three of the kids are brown-eyed, this implies that exactly one of them has blue eyes. Hence, we have :

$p_X(1) = \binom{n}{1} p^1 (1 - p)^{n-1} = \binom{4}{1} 0.25^1 \cdot 0.75^3 \approx 0.422.$
If five such families each have four kids what is the probability that two of them have exactly three blue eyed kids?

**Answer.** Now, if we define \( Y \) as the number of families that have exactly three blue-eyed kids, \( Y \) also follows a binomial distribution with parameters \( n_f = 5 \) (number of families) and \( p_f = p_X(3) \) (probability of having three blue-eyed kids).

\[
p_Y(2) = \binom{n_f}{2} p_f^2 (1 - p_f)^{n_f - 2} = \binom{5}{2} p_X(3)^2 (1 - p_X(3))^3 \approx 0.019.
\]

**Problem 4.** A student attends STAT394 three days a week. Assume that he oversleeps with probability 0.15.

(a) What is the probability the he misses one class in a week?

**Answer.** Let \( X_{\text{week}} \) be the number of classes the student misses in one week. Assuming that the underlying process is Bernoulli (namely independence and stationarity), \( X_{\text{week}} \) follows a binomial distribution \( \text{Bin}(n_{\text{week}}, p) \) with \( n_{\text{week}} = 3 \) and \( p = 0.15 \), the probability that he oversleeps. Therefore, the probability the he misses one class in a week is

\[
P(X_{\text{week}} = 1) = \binom{n_{\text{week}}}{1} p^1 (1 - p)^{n_{\text{week}} - 1} = \binom{3}{1} 0.15^1 0.85^2 \approx 0.325.
\]

(b) What is the probability the he misses three classes in a month (12 classes)?

**Answer.** Let \( X_{\text{month}} \) be the number of classes the student misses in one month. \( X_{\text{month}} \) is a binomial random variable with distribution parameters \( n_{\text{month}} = 12 \) and \( p = 0.15 \). Therefore, the probability the he misses three classes in a month is

\[
P(X_{\text{month}} = 3) = \binom{n_{\text{month}}}{3} p^3 (1 - p)^{n_{\text{month}} - 3} = \binom{12}{3} 0.15^3 0.85^9 \approx 0.172.
\]

(c) What is the probability the he misses at least two classes in a month?

**Answer.**

\[
P(X_{\text{month}} \geq 2) = 1 - P(X < 2)
= 1 - P(X \leq 1)
= 1 - (p_X(0) + p_X(1))
= 1 - \left( \begin{array}{c} 12 \\ 0 \end{array} \right) 0.15^0 0.85^{12} - \left( \begin{array}{c} 12 \\ 1 \end{array} \right) 0.15^1 0.85^{11}
\approx 0.557.
\]
(d) What is the probability the he misses four classes in total in a given month if he already missed two classes in that same month?

**Answer.** Conditioned on the fact he already missed two classes in the month, he is likely to miss four classes in that month with the following probability:

\[
P(X_{\text{month}} = 4 | X_{\text{month}} \geq 2) = \frac{P(\{X_{\text{month}} \geq 2\} \cap \{X_{\text{month}} = 4\})}{P(X_{\text{month}} \geq 2)}
\]

\[
= \frac{\binom{12}{4}0.15^40.85^8}{1 - \binom{12}{0}0.15^00.85^{12} - \binom{12}{1}0.15^10.85^{11}}
\]

\[
\approx 0.123.
\]

(e) How many classes does the instructor of STAT394 expect that student to miss at the end of the quarter (30 classes)? What is the corresponding variance?

**Answer.** Let \(X_{\text{quarter}}\) denote the number of classes the student misses during the quarter. Then \(X_{\text{quarter}}\) follows a binomial distribution \(Bin(n_{\text{quarter}}, p)\) with \(n_{\text{quarter}} = 30\) and \(p = 0.15\). The first part of the answer is given by the calculation of the expectation of \(X_{\text{quarter}}:\)

\[
E[X_{\text{quarter}}] = n_{\text{quarter}}p = 30 \cdot 0.15 = 4.5,
\]

and the corresponding variance is

\[
\text{Var}(X_{\text{quarter}}) = n_{\text{quarter}}p(1 - p) = 30 \cdot 0.15 \cdot 0.85 = 3.825.
\]