Problem 1. The probability mass function $p_X$ of a discrete random variable $X$ is given by the following table:

<table>
<thead>
<tr>
<th>$x$</th>
<th>$p_X(x)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>-1</td>
<td>0.25</td>
</tr>
<tr>
<td>0</td>
<td>0.2</td>
</tr>
<tr>
<td>1.5</td>
<td>0.25</td>
</tr>
<tr>
<td>2</td>
<td>0.1</td>
</tr>
<tr>
<td>2.5</td>
<td></td>
</tr>
</tbody>
</table>

(a) Give the missing value $p_X(-1)$.

Answer. Since $p_X(-1) + p_X(0) + p_X(1.5) + p_X(2) + p_X(2.5) = 1$, we have that

$$p_X(-1) = 1 - (0.25 + 0.2 + 0.25 + 0.1) = 0.2.$$ 

We notice that $p_X$ is a valid probability mass function since $p_X$ is nonnegative.

(b) The cumulative distribution function $F_X$ of $X$ is given by:

$$F_X(x) = \begin{cases} 
0 & x < -1 \\
a & -1 \leq x < 0 \\
0.45 & 0 \leq x < 1.5 \\
0.65 & 1.5 \leq x < 2 \\
b & 2 \leq x < 2.5 \\
1 & x \geq 2.5 
\end{cases}$$

Determine the constants $a$ and $b$.

Answer. The constants $a$ and $b$ can be obtained as follows:

$$a = F_X(-1) = p_X(-1) = 0.2,$$

and

$$b = F_X(2) = F_X(1.5) + p_X(2) = 0.65 + 0.25 = 0.9.$$ 

Problem 2. Determine the constant $c$ so that the following function $p$ satisfies the conditions of being a valid probability mass function: for $x \in \mathbb{N}$, $x \geq 1$, $p(x) = c \cdot (1/2)^x$.

Answer. The first condition requires that for $x \in \mathbb{N}$, $x \geq 1$, $p(x) \geq 0$. This clearly implies that $c > 0$. The second condition states that the values of $p$ should sum to 1:

$$\sum_{x=1}^{\infty} p(x) = 1 \iff c \sum_{x=1}^{\infty} (1/2)^x = 1$$

$$\iff c \cdot \frac{1/2}{1 - 1/2} = 1$$

$$\iff c = 1$$

We conclude that $c = 1$ is the only value that makes $p$ a valid probability mass function.
Problem 3. A person enters a bakery. The customer has 4 options: (s)he purchases

- nothing
- exactly one baguette
- exactly one croissant
- exactly one baguette and one croissant.

Assume that the customer buys a baguette with probability 0.7 and independently of whether or not (s)he buys a baguette, that person may pick a croissant with probability 0.4. A baguette costs $2.00 and a croissant $1.50. Let $X$ be the total dollar value spent by the customer.

(a) What are the possible values of $X$?

Answer. $X(\Omega) = \{0, 1.5, 2, 3.5\}$.

(b) Determine the probability mass function $p_X$ of $X$.

Answer. The corresponding probabilities are given by

- $p_X(0) = (1 - 0.7)(1 - 0.4) = 0.18$
- $p_X(1.5) = (1 - 0.7) \cdot 0.4 = 0.12$
- $p_X(2) = 0.7 \cdot (1 - 0.4) = 0.42$
- $p_X(3.5) = 0.7 \cdot 0.4 = 0.28$