STAT/MATH 395–PROBABILITY II
Distribution of Random Samples & Limit Theorems

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Outline

Distribution of i.i.d. Samples

Convergence of random variables

The Laws of Large Numbers

The Central Limit Theorem
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The Laws of Large Numbers

The Central Limit Theorem
Time spent on selected platforms by digital population in the United States as of December 2015 (in billion minutes)
Definition (i.i.d. Sample)

Let $X_1, \ldots, X_n$ be collection of $n \in \mathbb{N}^*$ random variables on probability space $(\Omega, \mathcal{A}, \mathbb{P})$. $(X_1, \ldots, X_n)$ is a random sample of size $n$ if and only if:

1. $X_1, \ldots, X_n$ are mutually independent
2. $X_1, \ldots, X_n$ are identically distributed

We say that $X_1, \ldots, X_n$ are independent and identically distributed (abbreviated i.i.d.).

Definition (Sample mean)

Let $(X_1, \ldots, X_n)$ be a random sample. Then the random variable

$$\bar{X}_n = \frac{1}{n} \sum_{i=1}^{n} X_i$$

is called the sample mean.
Definition (Independence)

Let $X_1, \ldots, X_n$ be $n \in \mathbb{N}^*$ rrvs on probability space $(\Omega, \mathcal{A}, \mathbb{P})$. Then $X_1, \ldots, X_n$ are said to be independent if and only if:

- (Discrete case) For all $(x_1, \ldots, x_n) \in X_1(\Omega) \times \ldots \times X_n(\Omega)$

$$p_{X_1 \ldots X_n}(x_1, \ldots, x_n) = \prod_{i=1}^{n} p_{X_i}(x_i), \quad (2)$$

- (Continuous case) For all $(x_1, \ldots, x_n) \in \mathbb{R}^n$

$$f_{X_1 \ldots X_n}(x_1, \ldots, x_n) = \prod_{i=1}^{n} f_{X_i}(x_i), \quad (3)$$
Property (Distribution of iid sample)

Let $(X_1, \ldots, X_n)$ be a random sample of size $n \in \mathbb{N}^*$ on probability space $(\Omega, \mathcal{A}, P)$. Then

- (Discrete case) For all $(x_1, \ldots, x_n) \in X_1(\Omega) \times \ldots \times X_n(\Omega)$

  \[
  p_{X_1 \ldots X_n}(x_1, \ldots, x_n) = \prod_{i=1}^{n} p_{X_1}(x_i),
  \] (4)

- (Continuous case) For all $(x_1, \ldots, x_n) \in \mathbb{R}^n$

  \[
  f_{X_1 \ldots X_n}(x_1, \ldots, x_n) = \prod_{i=1}^{n} f_{X_1}(x_i),
  \] (5)

Example. Let $(X_1, \ldots, X_n)$ be a random sample of size $n \in \mathbb{N}^*$ from an exponential distribution with parameter $\lambda$. Give the joint pdf of the sample.
Definition (Expected Value)

Let $X_1, \ldots, X_n$ be $n \in \mathbb{N}^*$ rrvs on probability space $(\Omega, \mathcal{A}, \mathbb{P})$ and let $g : \mathbb{R}^n \to \mathbb{R}$. Then, the mathematical expectation of $g(X_1 \ldots X_n)$, if it exists, is:

- **(Discrete case)**

$$
\mathbb{E}[g(X_1 \ldots X_n)] = \sum_{x_1 \in X_1(\Omega)} \cdots \sum_{x_n \in X_n(\Omega)} g(x_1, \ldots, x_n) p_{X_1 \ldots X_n}(x_1, \ldots, x_n)
$$

- **(Continuous case)**

$$
\mathbb{E}[g(X_1 \ldots X_n)] = \int \cdots \int_{\mathbb{R}^n} g(x_1, \ldots, x_n) f_{X_1 \ldots X_n}(x_1, \ldots, x_n)
$$
Theorem
Let \( X_1, \ldots, X_n \) be \( n \in \mathbb{N}^* \) independent rrvs on probability space \((\Omega, \mathcal{A}, \mathbb{P})\) and let \( g_1, \ldots, g_n \) be \( n \) real-valued functions on \( \mathbb{R} \). Then,

\[
\mathbb{E}[g_1(X_1) \ldots g_n(X_n)] = \mathbb{E}[g_1(X_1)] \ldots \mathbb{E}[g_n(X_n)]
\]  

(8)

provided that the expectations exist.

Theorem (Variance of independent rrvs)
Let \( X_1, \ldots, X_n \) be \( n \in \mathbb{N}^* \) independent rrvs on probability space \((\Omega, \mathcal{A}, \mathbb{P})\). Then,

\[
\text{Var} \left( \sum_{i=1}^{n} X_i \right) = \sum_{i=1}^{n} \text{Var}(X_i)
\]  

(9)

provided that the variances exist.
Property

Let \((X_1, \ldots, X_n)\) be a random sample of size \(n \in \mathbb{N}^*\) on probability space \((\Omega, \mathcal{A}, \mathbb{P})\) with mean \(\mu = \mathbb{E}[X_1]\) and variance \(\sigma^2 = \text{Var}(X_1) < \infty\). Then

\[
\mathbb{E}[\bar{X}_n] = \mu
\]  \hspace{1cm} (10)

and

\[
\text{Var}(\bar{X}_n) = \frac{\sigma^2}{n}
\]  \hspace{1cm} (11)
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- Distribution of i.i.d. Samples
- Convergence of random variables
- The Laws of Large Numbers
- The Central Limit Theorem
Definition
Let \((X_n)_{n \in \mathbb{N}^*}\) be a sequence of rrvs on probability space \((\Omega, A, \mathbb{P})\) and \(X\) be a rrv on the same probability space. Sequence \((X_n)\) is said to **converge in probability** towards \(X\) if, for all \(\epsilon > 0\):

\[
\lim_{n \to \infty} \mathbb{P}(|X_n - X| > \epsilon) = 0
\]  

(12)

Convergence in probability is denoted as follows:

\[X_n \xrightarrow{P} X\]

Example Let \(X\) be a discrete rrv with pmf \(p_X\) defined by:

\[
p_X(x) = \begin{cases} 
1/3 & \text{if } x = 1 \\
2/3 & \text{if } x = 0 \\
0 & \text{otherwise}
\end{cases}
\]

and let \(X_n = (1 + \frac{1}{n})X\).
Definition
Let $\left( X_n \right)_{n \in \mathbb{N}^*}$ be a sequence of rrvs on probability space $(\Omega, \mathcal{A}, \mathbb{P})$ and $X$ be a rrv on the same probability space. Sequence $(X_n)$ is said to converge almost surely or almost everywhere or with probability 1 or strongly towards $X$ if:

$$\mathbb{P}\left( \lim_{n \to \infty} X_n = X \right) = 1 \quad (13)$$

Almost sure convergence is denoted as follows:

$$X_n \xrightarrow{a.s.} X$$
Definition
Let \((X_n)_{n \in \mathbb{N}^*}\) be a sequence of rrvs on probability space \((\Omega, \mathcal{A}, \mathbb{P})\). For any \(n\), the distribution function of \(X_n\) is denoted by \(F_n\). Let \(X\) be a rrv with distribution function \(F_X\). Sequence \((X_n)\) is said to converge in distribution or converge weakly towards \(X\) if:

\[
\lim_{n \to \infty} F_n(x) = F_X(x) \tag{14}
\]

for all \(x \in \mathbb{R}\) at which \(F_X\) is continuous.

Convergence in distribution is denoted as follows:

\[X_n \overset{D}{\to} X\]

Example. Let \((X_n)_{n \in \mathbb{N}^*}\) be a sequence of rrvs with cdf \(F_n\)

\[F_n(x) = \left( 1 - \left( 1 - \frac{1}{n} \right)^{nx} \right) 1_{(0, \infty)}(x)\]
Theorem

Let \( (X_n)_{n \in \mathbb{N}^*} \) be a sequence of rrvs on probability space \((\Omega, \mathcal{A}, \mathbb{P})\) with respective mgfs \(M_n\). Let \(X\) be a rrv with mgf \(M_X\). If the following holds:

\[
\lim_{n \to \infty} M_n(x) = M_X(x) \quad (15)
\]

for all \(x \in \mathbb{R}\) where \(M_n(x)\) and \(M_X(x)\) exist, then sequence \((X_n)\) converges in distribution to \(X\).
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Property (Markov’s Inequality)

Let $X$ be a rrv that takes only on nonnegative values. Then, for any $a > 0$, we have:

$$
P(X \geq a) \leq \frac{\mathbb{E}[X]}{a} \tag{16}
$$

Property (Bienaymé-Chebyshev’s Inequality)

Let $X$ be a rrv that has expectation and variance. Then, for any $\alpha > 0$, we have:

$$
P(|X - \mathbb{E}[X]| \geq \alpha) \leq \frac{\text{Var}(X)}{\alpha^2} \tag{17}
$$
Theorem (Weak law of large numbers)

Let $(X_n)_{n \in \mathbb{N}^*}$ be a sequence of i.i.d. rrvs, each having finite expectation. The weak law of large numbers (also called Khintchine’s law) states that the sample mean $\overline{X}_n$ converges in probability towards $\mathbb{E}[X_1]$, that is, for all $\epsilon > 0$:

$$\lim_{n \to \infty} \mathbb{P} \left( |\overline{X}_n - \mathbb{E}[X_1]| > \epsilon \right) = 0$$

(18)
Theorem (Strong law of large numbers)

Let \((X_n)_{n \in \mathbb{N}^*}\) be a sequence of i.i.d. rrvs, each having finite expectation. The **strong law of large numbers** (also called Kolmogorov’s strong law) states that the sample mean \(\overline{X}_n\) converges almost surely towards \(\mathbb{E}[X_1]\), that is:

\[
P \left( \lim_{n \to \infty} \overline{X}_n = \mathbb{E}[X_1] \right) = 1
\]  

(19)
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Property

Let $X_1, \ldots, X_n$ be $n \in \mathbb{N}^*$ independent rrvs on probability space $(\Omega, \mathcal{A}, P)$ with respective moment generating functions $M_1, \ldots, M_n$. Then the moment generating function of

$$S_n = \sum_{i=1}^{n} X_i$$

is:

$$M_{S_n}(x) = \prod_{i=1}^{n} M_i(x) \quad (20)$$

for all $x \in \mathbb{R}$ where $M_n(x)$ exist.
Corollary (Mgf of iid sample)

Let $(X_1, \ldots, X_n)$ be a random sample of size $n \in \mathbb{N}^*$ on probability space $(\Omega, \mathcal{A}, \mathbb{P})$ with moment generating function $M = M_{X_1}$. Then the moment generating function of

$$S_n = \sum_{i=1}^{n} X_i$$

is:

$$M_{S_n}(x) = (M(x))^n$$

for all $x \in \mathbb{R}$ where $M(x)$ exists.
Theorem (Central Limit Theorem)

Let \((X_n)_{n \in \mathbb{N}^*}\) be a sequence of i.i.d. rrvs, each having expectation \(\mathbb{E}[X_1] = \mu\) and finite variance \(\text{Var}(X_1) = \sigma^2 < \infty\). The **Central Limit Theorem** states that the sequence of variables \((Z_n)_{n \in \mathbb{N}^*}\) defined by:

\[
Z_n = \frac{\bar{X}_n - \mu}{\sqrt{\sigma^2/n}}
\]

converges in distribution towards \(Z\) following a standard normal distribution \(\mathcal{N}(0, 1)\), that is:

\[
\lim_{n \to \infty} F_{Z_n}(x) = \Phi(x), \quad \text{for all } x \in \mathbb{R} \quad (22)
\]
Example

Let \((X_1, \ldots, X_{15})\) be a random sample with probability density function:

\[
f(x) = \frac{3}{2} x^2 \mathbb{1}_{(-1,1)}(x)
\]

What is the approximate probability that the sample mean \(\bar{X}_{15}\) falls between -2/5 and 1/5?
Let \((X_n)\) be a sequence of i.i.d. Poisson random variables with mean 3. Estimate approximately how large \(n\) must be such that:

\[
\mathbb{P}\left(\left| -3 + \frac{1}{n} \sum_{i=1}^{n} X_i \right| > 0.1 \right) = 0.1.
\]