HW5: Bivariate Distributions (3) – Solutions

Problem 1. Consider the set of points $G$ in the two-dimensional plane defined as:
$$G = \{(x, y) \in \mathbb{Z}^2 \mid |x| + |y| \leq 2\}$$
Remember that $\mathbb{Z}$ is the set of all integers.

(a) Give the list of all points in $G$.

**Answer.** $G = \{(-2,0), (-1,-1), (-1,0), (-1,1), (0,-2), (0,0), (0,1), (0,2), (1,-1), (1,0), (1,1), (2,0)\}$

(b) Assume that we pick a point from $G$ uniformly at random. Give $p_{XY}$ the joint probability mass function of $X$ and $Y$.

**Answer.** Here, we assume a discrete uniform distribution over the 13 points in $G$. Therefore,
$$p_{XY}(x, y) = \frac{1}{13} \mathbb{1}_{G}(x, y)$$

(c) Give $p_X$ and $p_Y$ the respective marginal probability mass functions of $X$ and $Y$.

**Answer.** The marginals are given by the following table:

<table>
<thead>
<tr>
<th>$X$ \ $Y$</th>
<th>-2</th>
<th>-1</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>$p_X(x)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>-2</td>
<td>0</td>
<td>0</td>
<td>1/13</td>
<td>0</td>
<td>0</td>
<td>1/13</td>
</tr>
<tr>
<td>-1</td>
<td>0</td>
<td>1/13</td>
<td>1/13</td>
<td>1/13</td>
<td>0</td>
<td>3/13</td>
</tr>
<tr>
<td>0</td>
<td>1/13</td>
<td>1/13</td>
<td>1/13</td>
<td>1/13</td>
<td>1/13</td>
<td>5/13</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>1/13</td>
<td>1/13</td>
<td>1/13</td>
<td>0</td>
<td>3/13</td>
</tr>
<tr>
<td>2</td>
<td>0</td>
<td>0</td>
<td>1/13</td>
<td>0</td>
<td>0</td>
<td>1/13</td>
</tr>
</tbody>
</table>

$py(y)$ = 1/13 3/13 5/13 3/13 1/13

(d) Give $p_{X|Y}(x|1)$ the conditional probability mass function of $X$ given $Y = 1$.

**Answer.** $p_{X|Y}(x|1)$ is given by:
$$p_{X|Y}(x|1) = \frac{p_{XY}(x,1)}{py(1)} = \frac{p_{XY}(x,1)}{3/13},$$

The values of $p_{X|Y}(x|1)$ can be read as follows:
(e) Are \(X\) and \(Y\) independent?

**Answer.** No since \(p_{X|Y}(1|1) = \frac{1}{3} \neq \frac{3}{13} = p_X(1)\).

(f) Find \(E[X|Y = 1]\) the expected value of \(X\) given \(Y = 1\).

**Answer.**

\[
E[X|Y = 1] = \sum_{x=-2}^{2} x p_{X|Y}(x|1)
= -2 \cdot 0 + (-1) \cdot \frac{1}{3} + 0 \cdot \frac{1}{3} + 1 \cdot \frac{1}{3} + 2 \cdot 0
= 0
\]

**Problem 2.** The random variables \(X\) and \(Y\) have a joint probability density function given by :
\[
f_{XY}(x, y) = 2e^{-(b(x+y))} \mathbf{1}_{S}(x, y)
\]
with \(S = \{(x, y) | 0 < x < y\}\).

(a) Show that the only value of \(b\) that makes \(f_{XY}\) a valid joint probability density function is \(b = 1\). A necessary condition for \(f_{XY}\) to be a valid joint probability density function is :

\[
\int_{0}^{\infty} \int_{x}^{\infty} 2e^{-(b(x+y))} \, dy \, dx = 1 \leftrightarrow \int_{0}^{\infty} e^{-bx} \left( \int_{x}^{\infty} e^{-by} \, dy \right) \, dx = \frac{1}{2}
\]

\[
\leftrightarrow \int_{0}^{\infty} e^{-bx} \left[ \frac{e^{-by}}{b} \right]_{x}^{\infty} \, dx = \frac{1}{2}
\]

\[
\leftrightarrow \int_{0}^{\infty} e^{-bx} \, dx = \frac{1}{2}
\]

\[
\leftrightarrow \frac{1}{b} \left[ -e^{-bx} \right]_{0}^{\infty} = \frac{1}{2}
\]

\[
\leftrightarrow \frac{1}{2b^2} = \frac{1}{2}
\]

\[
\leftrightarrow b^2 = 1
\]

The last equation leads to \(b\) is either -1 or 1. However the integral does not converge if \(b = -1\). Therefore there is only one value of \(b\) that makes \(f_{XY}\) a valid joint probability density function :

\[
b = 1
\]
(b) Compute \( f_X \) and \( f_Y \) the respective marginal probability density functions of \( X \) and \( Y \).

**Answer.** The marginal probability density function of \( X \) is given by: for \( x \in (0, \infty) \)

\[
  f_X(x) = \int_x^\infty 2e^{-(x+y)} \, dy \\
  = 2e^{-x} \int_x^\infty e^{-y} \, dy \\
  = 2e^{-x} \left[ -e^{-y} \right]_x^\infty 
\]

Hence,
\[
  f_X(x) = 2e^{-2x} \mathbf{1}_{(0, \infty)}(x). 
\]

The marginal probability density function of \( Y \) is given by: for \( y \in (0, \infty) \)

\[
  f_Y(y) = \int_0^y 2e^{-(x+y)} \, dx \\
  = 2e^{-y} \int_0^y e^{-x} \, dx \\
  = 2e^{-y} \left[ -e^{-x} \right]_0^y 
\]

Hence,
\[
  f_Y(y) = 2e^{-y} (1 - e^{-y}) \mathbf{1}_{(0, \infty)}(y). 
\]

(c) Are \( X \) and \( Y \) independent?

**Answer.** No since, for instance \( f_{XY}(1, 2) = 2e^{-3} \neq f_X(1)f_Y(2) = 2e^{-2} \cdot 2e^{-2} (1 - e^{-2}) \).

(d) If \( x > 0 \), compute \( f_{Y|X}(y|x) \) the conditional probability density function of \( Y \) given \( X = x \).

**Answer.** \( f_{Y|X}(y|x) \) is given by: for \( x > 0 \)

\[
  f_{Y|X}(y|x) = \frac{2e^{-(x+y)} \mathbf{1}_{(x, \infty)}(y)}{2e^{-2x}} \\
  = e^{-(y-x)} \mathbf{1}_{(x, \infty)}(y) 
\]

(e) If \( x > 0 \), compute \( \mathbb{E}[Y|X = x] \) the expected value of \( Y \) given \( X = x \).
Answer.

\[ E[Y|X = x] = \int_{-\infty}^{\infty} y f_{Y|X}(y|x) \, dy \]
\[ = \int_{x}^{\infty} ye^{-(y-x)} \, dy \]
\[ = e^x \left\{ [-ye^{-y}]_{x}^{\infty} - \int_{x}^{\infty} -e^{-y} \, dy \right\} \]
\[ = x + 1 \]