HW6: Distributions of Functions of Random Variables / Distribution of Random Samples & Limit theorems

Directions. Show and explain all work to receive full credit. Homework is due on Friday, March 3rd at the beginning of class by 10:00am.

Problem 1. Let $X$ with probability density function $f_X$ given by:

$$f_X(x) = \frac{x}{2} \mathbb{1}_{[0,2)}(x)$$

and let

$$Y = 1 - \sqrt{4 - X^2}.$$

Find $f_Y(y)$ the probability density function of $Y$. The answer must include the support of $f_Y$.

Problem 2. Let $X$ be a standard normal random variable $\mathcal{N}(0, 1)$ and let $Y = X^2$. Find $f_Y(y)$ the probability density function of $Y$. The answer must include the support of $f_Y$.

Problem 3. Let $X$ and $Y$ with joint probability density function $f_{XY}$ given by:

$$f_{XY}(x, y) = xe^{-(x+y)} \mathbb{1}_{(0,\infty)^2}(x, y).$$

Find $f_{WZ}$ the joint probability density function of $W = X + Y$ and $Z = Y/X$. The answer must include the support of $f_{WZ}$.

Problem 4. Let $X$ and $Y$ with joint probability density function $f_{XY}$ given by:

$$f_{XY}(x, y) = \mathbb{1}_{[0,1]^2}(x, y)$$

and let $W = X/Y$ and $Z = X + Y$. Find $f_{WZ}(w, z)$ the joint probability density function of $W$ and $Z$. The answer must include the support of $f_{WZ}$.

Problem 5. Let $X$ be a continuous random variable on a probability space $(\Omega, \mathcal{A}, P)$. For $n \geq 1$, let $(X_n)$ and $(Y_n)$ be two sequences of continuous random variables on the same probability space such that $X_n = X + Y_n$, where

$$\mathbb{E}[Y_n] = \frac{1}{n} \text{ and } \text{Var}(Y_n) = \frac{\sigma^2}{n},$$

where $\sigma > 0$ is a constant. The objective is to show that the sequence $(X_n)$ converges in probability to $X$. 


(a) Let $W$ be a continuous random variable with probability density function $f_W$. Show that for any positive constants $a$ and $\varepsilon$,

$$P(|W - a| + a > \varepsilon) \geq P(|W| > \varepsilon).$$

**Hint.** Show that for $\varepsilon > a$, 

$$P(|W - a| + a > \varepsilon) - P(|W| > \varepsilon) = \int_{-\varepsilon}^{2a-\varepsilon} f_W(w) \, dw.$$

(b) Derive from (a) that for any $\varepsilon > 0$,

$$P(|X_n - X| > \varepsilon) \leq P\left(|Y_n - E[Y_n]| > \varepsilon - \frac{1}{n}\right)$$

(c) Use Chebyshev’s inequality to conclude that $(X_n)$ converges in probability to $X$. 