Outline

Distributions of Two Random Variables
  Distributions of Two Discrete Random Variables
  Distributions of Two Continuous Random Variables

The Correlation Coefficient
  Covariance
  Correlation

Conditional Distributions
  Discrete case
  Continuous case
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Example: Climate Change

Mapping the Impacts of Climate Change

[Map showing the impacts of climate change across various regions, with different colors representing varying levels of impact.]

[Graph illustrating a network of nodes labeled with terms such as H2, CO2, VAP, AER, etc., connected by arrows indicating relationships or flows.]
Objectives

- Extend the definition of a probability distribution of one random variable to the **joint** probability distribution of two random variables
- Learn a way of quantifying the extent to which two random variables are related
- Define the conditional probability distribution of a random variable $X$ given that $Y$ has occurred
Definition
Let $X$ and $Y$ be two rrvs on probability space $(\Omega, \mathcal{A}, \mathbb{P})$. A **two-dimensional random variable** $(X, Y)$ is a function mapping $(X, Y): \Omega \rightarrow \mathbb{R}^2$, such that for any numbers $x, y \in \mathbb{R}$:

$$\{\omega \in \Omega \mid X(\omega) \leq x \text{ and } Y(\omega) \leq y\} \in \mathcal{A}$$  \hspace{1cm} (1)

Definition (Joint cumulative distribution function)
Let $X$ and $Y$ be two rrvs on probability space $(\Omega, \mathcal{A}, \mathbb{P})$. The **joint (cumulative) distribution function** (joint c.d.f.) of $X$ and $Y$ is the function $F_{XY}$ given by

$$F_{XY}(x, y) = \mathbb{P}(\{X \leq x\} \cap \{Y \leq y\}) \triangleq \mathbb{P}(X \leq x, Y \leq y),$$  \hspace{1cm} (2)

for $x, y \in \mathbb{R}$
Why Gen Y Loves Restaurants – And Restaurants Love Them Even More

According to a new report from the research firm Technomic, 42% of millennials say they visit “upscale casual-dining restaurants” at least once a month. That’s a higher percentage than Gen X (33%) and Baby Boomers (24%) who go to such restaurants once or more monthly.

Time, Aug. 15, 2012, By Brad Tuttle

<table>
<thead>
<tr>
<th>Age</th>
<th>Population</th>
</tr>
</thead>
<tbody>
<tr>
<td>0-19</td>
<td>83,267,556</td>
</tr>
<tr>
<td>20-34 (Millenials)</td>
<td>62,649,947</td>
</tr>
<tr>
<td>35-49 (Gen X)</td>
<td>63,779,197</td>
</tr>
<tr>
<td>50-69 (Baby Boomers)</td>
<td>71,216,117</td>
</tr>
<tr>
<td>70+</td>
<td>27,832,721</td>
</tr>
</tbody>
</table>

Demography in the US by age group [US Census data]
Definition

Let $X$ and $Y$ be two discrete rrvs on probability space $(\Omega, \mathcal{A}, \mathbb{P})$. The **joint pmf** of $X$ and $Y$, denoted by $p_{XY}$, is defined as follows:

$$p_{XY}(x, y) = \mathbb{P} \left( \{X = x\} \cap \{Y = y\} \right) \triangleq \mathbb{P}(X = x, Y = y), \quad (3)$$

for $x \in X(\Omega)$ and $y \in Y(\Omega)$.
Why Gen Y Loves Restaurants – And Restaurants Love Them Even More

- $X = 1$ : the person visits upscale restaurants at least once a month / $\{X = 0\} = \{X = 1\}^c$
- $Y = 1$ : the person is a millenial / $Y = 2$ : the person is a Gen X / $Y = 3$ : the person is a Baby Boomer

<table>
<thead>
<tr>
<th>$X$</th>
<th>$Y$</th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>36,337</td>
<td>42,732</td>
<td>47,003</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>26,313</td>
<td>21,047</td>
<td>24,213</td>
<td></td>
</tr>
</tbody>
</table>

Table 1: Count Data ($\times1000$)

<table>
<thead>
<tr>
<th>$X$</th>
<th>$Y$</th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0.184</td>
<td>0.216</td>
<td>0.238</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>0.133</td>
<td>0.106</td>
<td>0.123</td>
<td></td>
</tr>
</tbody>
</table>

Table 2: Joint Probability Mass Function $p_{XY}(x,y)$
Property

Let $X$ and $Y$ be two discrete rrvs on probability space $(\Omega, \mathcal{A}, \mathbb{P})$ with joint pmf $p_{XY}$, then the following holds:

1. $p_{XY}(x, y) \geq 0$, for $x \in X(\Omega)$ and $y \in Y(\Omega)$
2. $\sum_{x \in X(\Omega)} \sum_{y \in Y(\Omega)} p_{XY}(x, y) = 1$

Property

Let $X$ and $Y$ be two discrete rrvs on probability space $(\Omega, \mathcal{A}, \mathbb{P})$ with joint pmf $p_{XY}$. Then, for any subset $B \subset X(\Omega) \times Y(\Omega)$

$$\mathbb{P}((X, Y) \in B) = \sum_{(x,y)\in B} p_{XY}(x, y)$$
Definition (Marginal distributions)

Let $X$ and $Y$ be two discrete rrvs on probability space $(\Omega, \mathcal{A}, \mathbb{P})$ with joint pmf $p_{XY}$. Then the pmf of $X$ alone is called the **marginal probability mass function** of $X$ and is defined by:

$$p_X(x) = \mathbb{P}(X = x) = \sum_{y \in Y(\Omega)} p_{XY}(x, y), \quad \text{for } x \in X(\Omega) \quad (4)$$

Similarly, the pmf of $Y$ alone is called the **marginal probability mass function** of $Y$ and is defined by:

$$p_Y(y) = \mathbb{P}(Y = y) = \sum_{x \in X(\Omega)} p_{XY}(x, y), \quad \text{for } y \in Y(\Omega) \quad (5)$$
Definition (Independence)

Let $X$ and $Y$ be two discrete rrvs on probability space $(\Omega, \mathcal{A}, \mathbb{P})$ with joint pmf $p_{XY}$. Let $p_X$ and $p_Y$ be the respective marginal pmfs of $X$ and $Y$. Then $X$ and $Y$ are said to be independent if and only if:

$$p_{XY}(x, y) = p_X(x)p_Y(y), \quad \text{for all } x \in X(\Omega) \text{ and } y \in Y(\Omega) \quad (6)$$
Definition (Expected Value)

Let $X$ and $Y$ be two discrete rrvs on probability space $(\Omega, \mathcal{A}, \mathbb{P})$ with joint pmf $p_{XY}$ and let $g : \mathbb{R}^2 \rightarrow \mathbb{R}$ be a bounded piecewise continuous function. Then, the mathematical expectation of $g(X, Y)$, if it exists, is denoted by $\mathbb{E}[g(X, Y)]$ and is defined as follows:

$$
\mathbb{E}[g(X, Y)] = \sum_{x \in X(\Omega)} \sum_{y \in Y(\Omega)} g(x, y) p_{XY}(x, y) \quad (7)
$$
Example

Consider the following joint probability mass function:

\[ p_{XY}(x, y) = \frac{xy^2}{13} \mathbb{1}_S(x, y) \]

with \( S = \{(1, 1), (1, 2), (2, 2)\} \)

1. Show that \( p_{XY} \) is a valid joint probability mass function.
2. What is \( P(X + Y \leq 3) \)?
3. Give the marginal probability mass functions of \( X \) and \( Y \).
4. What are the expected values of \( X \) and \( Y \)?
5. Are \( X \) and \( Y \) independent?
**Definition (Joint probability density function)**

Let $X$ and $Y$ be two continuous rrvs on probability space $(\Omega, \mathcal{A}, \mathbb{P})$. $X$ and $Y$ are said to be **jointly continuous** if there exists a function $f_{XY}$ such that, for any Borel set on $\mathbb{R}^2$:

$$
\mathbb{P}((X, Y) \in B) = \int \int_B f_{XY}(x, y) \, dx \, dy
$$

(8)

Then function $f_{XY}$ is called the **joint probability density function** of $X$ and $Y$.

Moreover, if the joint distribution function $F_{XY}$ is of class $C^2$, then the joint pdf of $X$ and $Y$ can be expressed in terms of partial derivatives:

$$
f_{XY}(x, y) = \frac{\partial^2 F(x, y)}{\partial x \partial y}
$$

(9)
Property

Let $X$ and $Y$ be two continuous rrvs on probability space $(\Omega, \mathcal{A}, \mathbb{P})$ with joint pdf $f_{XY}$, then the following holds:

$\begin{itemize}
  \item f_{XY}(x, y) \geq 0, \text{ for } x, y \in \mathbb{R}
  \item \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_{XY}(x, y) \, dx \, dy = 1
\end{itemize}$

Example. Let $X$ and $Y$ be two continuous random variables with joint probability density function:

$f_{XY}(x, y) = 4xy \mathbb{1}_{[0,1]^2}(x,y)$

$\begin{itemize}
  \item Verify that $f_{XY}$ is a valid joint probability density function.
  \item What is $\mathbb{P}(Y < X)$?
\end{itemize}$
Definition (Marginal distributions)

Let $X$ and $Y$ be two continuous rrvs on probability space $(\Omega, \mathcal{A}, P)$ with joint pdf $f_{XY}$. Then the pdf of $X$ alone is called the **marginal probability density function** of $X$ and is defined by:

$$f_X(x) = \int_{-\infty}^{\infty} f_{XY}(x, y) \, dy, \quad \text{for } x \in \mathbb{R} \quad (10)$$

Similarly, the pdf of $Y$ alone is called the **marginal probability density function** of $Y$ and is defined by:

$$f_Y(y) = \int_{-\infty}^{\infty} f_{XY}(x, y) \, dx, \quad \text{for } y \in \mathbb{R} \quad (11)$$

**Example.** Let $X$ and $Y$ be two continuous random variables with joint probability density function :

$$f_{XY}(x, y) = 4xy 1_{[0,1]^2}(x, y)$$

What are the marginal probability density functions of $X$ and $Y$?
Definition (Independence)

Let $X$ and $Y$ be two continuous random variables on probability space $(\Omega, \mathcal{A}, \mathbb{P})$ with joint pdf $f_{XY}$. Let $f_X$ and $f_Y$ be the respective marginal pdfs of $X$ and $Y$. Then $X$ and $Y$ are said to be independent if and only if:

$$f_{XY}(x, y) = f_X(x)f_Y(y), \quad \text{for all } x, y \in \mathbb{R}$$

(12)

Example. Let $X$ and $Y$ be two continuous random variables with joint probability density function:

$$f_{XY}(x, y) = 4xy \mathbb{1}_{[0,1]^2}(x, y)$$

Are $X$ and $Y$ independent?
Definition (Expected Value)

Let $X$ and $Y$ be two continuous rrvs on probability space $(\Omega, \mathcal{A}, P)$ with joint pdf $f_{XY}$ and let $g : \mathbb{R}^2 \rightarrow \mathbb{R}$ be a bounded piecewise continuous function. Then, the mathematical expectation of $g(X, Y)$, if it exists, is denoted by $\mathbb{E}[g(X, Y)]$ and is defined as follows:

$$\mathbb{E}[g(X, Y)] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(x, y) f_{XY}(x, y) \, dx \, dy \quad (13)$$

Example. Let $X$ and $Y$ be two continuous random variables with joint probability density function:

$$f_{XY}(x, y) = 4xy 1_{[0,1]^2}(x, y)$$

What are the expected values of $X$ and $Y$?
Example

The joint probability density function of $X$ and $Y$ is given by:

$$f_{XY}(x, y) = \frac{6}{7} \left( x^2 + \frac{xy}{2} \right) 1_S(x, y)$$

with $S = \{(x, y) | 0 < x < 1, 0 < y < 2\}$

1. Show that $f_{XY}$ is a valid joint probability density function.
2. Compute the marginal probability density function of $X$.
3. Find $\mathbb{P}(X > Y)$.
4. What are the expected values of $X$ and $Y$?
5. Are $X$ and $Y$ independent?
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Definition (Covariance)

Let $X$ and $Y$ be two rrvs on probability space $(\Omega, \mathcal{A}, \mathbb{P})$. The **covariance** of $X$ and $Y$, denoted by $\text{Cov}(X, Y)$, is defined as follows:

$$\text{Cov}(X, Y) = \mathbb{E}[(X - \mathbb{E}[X])(Y - \mathbb{E}[Y])]$$  \hspace{1cm} (14)

upon existence of the above expression.

- If $X$ and $Y$ are discrete rrv with joint pmf $p_{XY}$, then the covariance of $X$ and $Y$ is:

$$\text{Cov}(X, Y) = \sum_{x \in X(\Omega)} \sum_{y \in Y(\Omega)} (x - \mathbb{E}[X])(y - \mathbb{E}[Y])p_{XY}(x, y)$$ \hspace{1cm} (15)

- If $X$ and $Y$ are continuous rrv with joint pdf $f_{XY}$, then the covariance of $X$ and $Y$ is:

$$\text{Cov}(X, Y) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (x - \mathbb{E}[X])(y - \mathbb{E}[Y])f_{XY}(x, y) \, dx \, dy$$ \hspace{1cm} (16)
Theorem
Let $X$ and $Y$ be two rrvs on probability space $(\Omega, \mathcal{A}, \mathbb{P})$. The covariance of $X$ and $Y$ can be calculated as:

$$\text{Cov}(X, Y) = \mathbb{E}[XY] - \mathbb{E}[X] \mathbb{E}[Y]$$  \hspace{1cm} (17)$$

Example 6. Suppose that $X$ and $Y$ have the following joint probability mass function:

<table>
<thead>
<tr>
<th>(X) (Y)</th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.25</td>
<td>0.25</td>
<td>0</td>
</tr>
<tr>
<td>2</td>
<td>0</td>
<td>0.25</td>
<td>0.25</td>
</tr>
</tbody>
</table>

What is the covariance of $X$ and $Y$?
Property

Here are some properties of the covariance. For any random variables $X$ and $Y$, we have:

1. $\text{Cov}(X, Y) = \text{Cov}(Y, X)$
2. $\text{Cov}(X, X) = \text{Var}(X)$
3. $\text{Cov}(aX, Y) = a\text{Cov}(X, Y)$ for $a \in \mathbb{R}$
4. Let $X_1, \ldots, X_n$ be $n$ random variables and $Y_1, \ldots, Y_m$ be $m$ random variables. Then:

$$
\text{Cov} \left( \sum_{i=1}^{n} X_i, \sum_{j=1}^{m} Y_j \right) = \sum_{i=1}^{n} \sum_{j=1}^{m} \text{Cov}(X_i, Y_j)
$$
Definition (Correlation)

Let $X$ and $Y$ be two rrvs on probability space $(\Omega, \mathcal{A}, \mathbb{P})$ with respective standard deviations $\sigma_X = \sqrt{\text{Var}(X)}$ and $\sigma_Y = \sqrt{\text{Var}(Y)}$. The correlation of $X$ and $Y$, denoted by $\rho_{XY}$, is defined as follows:

$$\rho_{XY} = \frac{\text{Cov}(X, Y)}{\sigma_X \sigma_Y}$$  \hspace{1cm} (18)

upon existence of the above expression

Property

Let $X$ and $Y$ be two rrvs on probability space $(\Omega, \mathcal{A}, \mathbb{P})$.

$$-1 \leq \rho_{XY} \leq 1$$  \hspace{1cm} (19)
Interpretation of Correlation

The correlation coefficient of $X$ and $Y$ is interpreted as follows:

- If $\rho_{XY} = 1$, then $X$ and $Y$ are perfectly, positively, linearly correlated.
- If $\rho_{XY} = -1$, then $X$ and $Y$ are perfectly, negatively, linearly correlated. $X$ and $Y$ are also said to be perfectly linearly anticorrelated.
- If $\rho_{XY} = 0$, then $X$ and $Y$ are completely, linearly uncorrelated.
- If $0 < \rho_{XY} < 1$, then $X$ and $Y$ are positively, linearly correlated.
- If $-1 < \rho_{XY} < 0$, then $X$ and $Y$ are negatively, linearly correlated.
Example: Dining Habits

- $X = 1$: a person between 20 and 69 visits upscale restaurants at least once a month and $X = 0$ otherwise
- $Y = 1$: a person between 20 and 69 is a millenial
- $Y = 2$: a person between 20 and 69 is a Gen X
- $Y = 3$: a person between 20 and 69 is a Baby Boomer

The joint probability mass function of $X$ and $Y$ is given below:

<table>
<thead>
<tr>
<th></th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0.184</td>
<td>0.216</td>
<td>0.238</td>
</tr>
<tr>
<td>1</td>
<td>0.133</td>
<td>0.106</td>
<td>0.123</td>
</tr>
</tbody>
</table>

Compute the covariance and the correlation of $X$ and $Y$. What can you say about the relationship between $X$ and $Y$?
Correlation is not causation!

Number of people who drowned by falling into a pool correlates with Films Nicolas Cage appeared in

Correlation: 66.6% (r=0.666004)
Theorem
Let $X$ and $Y$ be two rrvs on probability space $(\Omega, \mathcal{A}, \mathbb{P})$ and let $g_1$ and $g_2$ be two bounded piecewise continuous functions. If $X$ and $Y$ are independent, then

$$
\mathbb{E}[g_1(X)g_2(Y)] = \mathbb{E}[g_1(X)] \mathbb{E}[g_2(Y)]
$$

provided that the expectations exist.

Theorem
Let $X$ and $Y$ be two rrvs on probability space $(\Omega, \mathcal{A}, \mathbb{P})$. If $X$ and $Y$ are independent, then

$$
\text{Cov}(X, Y) = \rho_{XY} = 0
$$
Example

Let $X$ and $Y$ be discrete random variables with joint pmf:

<table>
<thead>
<tr>
<th>$X$ \ $Y$</th>
<th>-1</th>
<th>0</th>
<th>1</th>
</tr>
</thead>
<tbody>
<tr>
<td>-1</td>
<td>0.2</td>
<td>0</td>
<td>0.2</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
<td>0.2</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>0.2</td>
<td>0</td>
<td>0.2</td>
</tr>
</tbody>
</table>

1. What is the correlation between $X$ and $Y$?
2. Are $X$ and $Y$ independent?
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An international TV network company is interested in the relationship between the region of citizenship of its customers and their favorite sport.

<table>
<thead>
<tr>
<th>Sports</th>
<th>Citizenship</th>
<th>Africa</th>
<th>America</th>
<th>Asia</th>
<th>Europe</th>
</tr>
</thead>
<tbody>
<tr>
<td>Tennis</td>
<td></td>
<td>0.02</td>
<td>0.07</td>
<td>0.04</td>
<td>0.12</td>
</tr>
<tr>
<td>Basketball</td>
<td></td>
<td>0.03</td>
<td>0.11</td>
<td>0.01</td>
<td>0.04</td>
</tr>
<tr>
<td>Soccer</td>
<td></td>
<td>0.08</td>
<td>0.05</td>
<td>0.04</td>
<td>0.16</td>
</tr>
<tr>
<td>Football</td>
<td></td>
<td>0.01</td>
<td>0.17</td>
<td>0.02</td>
<td>0.03</td>
</tr>
</tbody>
</table>
**Definition**

Let $X$ and $Y$ be two discrete rrvs on probability space $(\Omega, \mathcal{A}, \mathbb{P})$ with joint pmf $p_{XY}$ and respective marginal pmfs $p_X$ and $p_Y$. Then, for $y \in Y(\Omega)$, the **conditional probability mass function** of $X$ given $Y = y$ is defined by:

$$p_{X|Y}(x|y) = \mathbb{P}(X = x|Y = y) = \frac{p_{XY}(x, y)}{p_Y(y)} \quad (22)$$

provided that $p_Y(y) \neq 0$. 
Definition (Conditional Expectation)

Let $X$ and $Y$ be two discrete rrvs on probability space $(\Omega, \mathcal{A}, \mathbb{P})$. Then, for $y \in Y(\Omega)$, the conditional expectation of $X$ given $Y = y$ is defined as follows:

$$
\mathbb{E}[X|Y = y] = \sum_{x \in X(\Omega)} x p_{X|Y}(x|y) \quad (23)
$$

Property (Linearity of conditional expectation)

Let $X$ and $Y$ be two rrvs on probability space $(\Omega, \mathcal{A}, \mathbb{P})$. If $c_1, c_2 \in \mathbb{R}$ and $g_1 : X(\Omega) \to \mathbb{R}$ and $g_2 : X(\Omega) \to \mathbb{R}$ are piecewise continuous functions. Then, for $y \in Y(\Omega)$, we have:

$$
\mathbb{E}[c_1 g_1(X) + c_2 g_2(X)|Y = y] = c_1 \mathbb{E}[g_1(X)|Y = y] + c_2 \mathbb{E}[g_2(X)|Y = y] \quad (24)
$$
Property

Let $X$ and $Y$ be two discrete rrvs on probability space $(\Omega, \mathcal{A}, \mathbb{P})$. If $X$ and $Y$ are independent, then, for $y \in Y(\Omega)$, we have:

$$p_{X|Y}(x|y) = p_X(x) \quad \text{for all } x \in X(\Omega)$$  \hfill (25)

Similarly, for $x \in X(\Omega)$, we have:

$$p_{Y|X}(y|x) = p_Y(y) \quad \text{for all } y \in Y(\Omega)$$  \hfill (26)

Also,

$$\mathbb{E}[X|Y = y] = \mathbb{E}[X]$$  \hfill (27)

Similarly, for $x \in X(\Omega)$, we have:

$$\mathbb{E}[Y|X = x] = \mathbb{E}[Y]$$  \hfill (28)

Remark: Those properties are also true for continuous rrvs.
Definition (Conditional Variance)

Let $X$ and $Y$ be two rrvs on probability space $(\Omega, \mathcal{A}, \mathbb{P})$. Then, for $y \in Y(\Omega)$, the **conditional variance** of $X$ given $Y = y$ is defined as follows:

$$
\text{Var}(X|Y = y) = \mathbb{E}[(X - \mathbb{E}[X|Y = y])^2|Y = y] \quad (29)
$$

Property

For $y \in Y(\Omega)$, the **conditional variance** of $X$ given $Y = y$ can be calculated as follows:

$$
\text{Var}(X|Y = y) = \mathbb{E}[X^2|Y = y] - (\mathbb{E}[X|Y = y])^2 \quad (30)
$$
Definition

Let $X$ and $Y$ be two continuous rrvs on probability space $(\Omega, \mathcal{A}, P)$ with joint pdf $f_{XY}$ and respective marginal pdfs $f_X$ and $f_Y$. Then, for $y \in \mathbb{R}$, the conditional probability density function of $X$ given $Y = y$ is defined by:

$$f_{X|Y}(x|y) = \frac{f_{XY}(x,y)}{f_Y(y)} \quad (31)$$

provided that $f_Y(y) \neq 0$. 
Definition (Conditional Expectation)

Let $X$ and $Y$ be two continuous rrvs on probability space $(\Omega, \mathcal{A}, \mathbb{P})$. Then, for $y \in \mathbb{R}$, the \textit{conditional expectation} of $X$ given $Y = y$ is defined as follows:

$$
\mathbb{E}[X|Y = y] = \int_{-\infty}^{\infty} xf_{X|Y}(x|y) \, dx
$$

(32)
Example

Let $X$ and $Y$ be two continuous random variables with joint probability density function:

$$f_{XY}(x, y) = \frac{3}{2} \mathbb{1}_S(x, y)$$

with $S = \{(x, y) | x^2 < y < 1, 0 < x < 1\}$

Compute the expected value of $Y$ given $X = x$. 
