Problem Set 2

**Problem 1.** Let $X$ and $Y$ be two random variables with respective standard deviations $\sigma_X$ and $\sigma_Y$ and with correlation coefficient $\rho_{XY}$.

(a) Show that:
\[
\text{Var}\left( \frac{X}{\sigma_X} + \frac{Y}{\sigma_Y} \right) = 2(1 + \rho_{XY})
\]

Hint: Use the fact that for any random variable $Z$, $\text{Var}(Z) = \text{Cov}(Z, Z)$ and that the covariance is a linear operator.

(b) Show that:
\[
\text{Var}\left( \frac{X}{\sigma_X} - \frac{Y}{\sigma_Y} \right) = 2(1 - \rho_{XY})
\]

(c) Derive from (a) and (b) that:
\[-1 \leq \rho_{XY} \leq 1\]

**Problem 2.** If $X$ and $Y$ have joint probability density function:
\[
f_{XY}(x, y) = \frac{1}{y} \mathbb{1}_S(x, y)
\]
with $S = \{(x, y) | 0 < y < 1, 0 < x < y\}$. Compute $\text{Cov}(X, Y)$.

**Problem 3.** An apartment manager decides to see if timely payment of rent is related to unpaid credit card balances. Using $X$ as the number of timely rent payments in the last four months and $Y$ as the number of unpaid credit card balances, he examines his records and the credit reports of his renters and obtains the following results.

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<th>$X$ \ $Y$</th>
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(a) Interpret in words the value $p_{XY}(1, 2)$.

(b) Fill in the table with the marginal probability mass function values of $X$ and $Y$.

(c) Find the proportion of renters that have at most 2 timely rent payments in the last four months.
(d) What is the probability that a randomly selected tenant has more unpaid credit card balances than timely rent payments in the last four months?

(e) What is the mean number of unpaid credit card balances?

(f) You are given: \( \mathbb{E}[X] = 2.31, \mathbb{E}[Y] = 1.37, \mathbb{E}[XY] = 2.61, \mathbb{E}[X^2] = 6.99 \) and \( \mathbb{E}[Y^2] = 2.87 \). Compute \( \rho_{XY} \) the correlation of \( X \) and \( Y \).

(g) You are given: \( \rho_{XY} \approx -0.43 \). What can you say about the relationship between the number of credit card balances and the timeliness of rent payments? Justify your answer.

**Problem 4.** The buildings in a given area are designed such that the temperature inside the building \( T_{in} \) (in kelvins) and the temperature outside the building \( T_{out} \) (in kelvins) have a joint probability density function given by:

\[
f_{T_{in},T_{out}}(x,y) = 2e^{-2x} \frac{y}{x} 1_S(x,y)
\]

with \( S = \{(x,y) | x \geq 0, 0 \leq y \leq x \} \).

(a) Compute \( f_{T_{in}} \) and \( f_{T_{out}} \) the respective marginal probability density functions of \( T_{in} \) and \( T_{out} \).

(b) Are \( T_{in} \) and \( T_{out} \) independent?

(c) Compute \( \text{Cov}(T_{in},T_{out}) \).

**Problem 5.**

(a) Let \( X \) be a real-valued random variable with moment generating function \( M_X \). Show that for any constants \( a, b \in \mathbb{R} \), the moment generating function of \( aX + b \) is

\[
M_{aX+b}(t) = e^{bt} M_X(at).
\]

(b) The dairy industry in some country is facing a major crisis. The price \( P \) in units of currency (uc) for one gallon of milk has moment generating function

\[
M_P(t) = \frac{1}{2} (1 + e^t).
\]

The government proposes to multiply the price by 1.5 and add 0.1 uc. Compute the moment generating function of the new price.

(c) Show that the expectation of the new price is 0.85 uc.

(d) Compute the variance of the new price.

(e) The dairy farmers claim that a fair price would rather be \( e^P \). Show that the expectation of the price demanded by the farmers is approximately 1.86 uc. Hint. Evaluate \( M_P(t) \) for a particular value of \( t \).

(f) Compute the variance of the price demanded by the farmers. Hint. Evaluate \( M_P(t) \) for two particular values of \( t \).